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Economic analysis in the context of incomplete knowledge

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TABLE OF CONTENTS

		Page
I.	INTRODUCTION	1
II.	LITERATURE REVIEW	5
	A. Decision Making under Uncertainty	6
	B. Decision Making under Risk	9
	C. Decision Making under Incomplete Knowledge	14
III.	EXTREME VARIANCES OF PAYOFFS UNDER STRICT RANKING	18
	A. Transformation of Objective Function and Constraints	23
	B. Nature of Transformed Objective Function and Constraints	28
	C. General Approach to Search for Extreme Variances	29
	D. Extreme Variances for Two States of Nature	31
	E. Extreme Variances for Three States of Nature	34
	F. Extreme Variances for N States of Nature	41
IV.	EXTREME INDEXES OF UTILITY UNDER STRICT RANKING	60
	A. Objective Function and Constraints	66
	B. Nature of Objective Function and Constraints	67
	C. Approach to Search for Extreme Indexes of Utility	68
	D. Extreme Indexes of Utility for Two States of Nature	70
	E. Extreme Indexes of Utility for N States of Nature	75
V.	DECISION PROCEDURE IN THE CONTEXT OF UNCERTAINTY AND RISK	97
	A. Nature of the Coefficient of Risk Aversion	97
	B. Constant's Decision Procedure in the Context of Uncertainty	98
	C. Application of Constant's Decision Procedure	102

	D. Simplified Approach for Constant's Decision Procedure	111
	E. Application of Simplified Approach to the Context of Risk	118
VI.	DECISION PROCEDURE FOR WEAK RANKING IN THE CONTEXT OF INCOMPLETE KNOWLEDGE	121
	A. Extreme Indexes of Utility under Weak Ranking	124
	B. Maximum and Minimum Utility Index Rates of Return	133
	C. Determination of Final Angular Coefficient	138
	D. Summary	142
VII.	DECISION PROCEDURE FOR STRICT RANKING IN THE CONTEXT OF INCOMPLETE KNOWLEDGE	151
	A. Extreme Values of Index of Utility	153
	B. Maximum and Minimum Utility Index Rates of Return	158
	C. Determination of Final Angular Coefficient	161
	D. Summary	164
VIII.	CONCLUSION	173
IX.	BIBLIOGRAPHY	176
X.	ACKNOWLEDGMENTS	179

I. INTRODUCTION

In today's business community, there is an increasing awareness of the increasing complexity in the evaluation of capital alternatives and this complexity has resulted in a growing concern for the risk imposed by the market place. As a result, economic decision procedures that recognize and reduce possible risks are increasingly needed. This research focuses on the treatment of multi-valued estimates in the evaluation of mutually exclusive alternatives.

Risk can be defined on a general level as the potential for realization of unwanted, negative consequence of an event [Rowe 1977]. However, it is important that the control of risks be considered as well.

Conventional economic decision models have involved only single valued estimates. When one considers the uncertainty that may be related to the array of variables that form the final figure of merit in an economic decision, i.e., rate of return or present worth, one is tempted to question the purely deterministic approach that is habitually followed to solve this type of problem.

The multi-valued estimate problem has been traditionally addressed in the context of risk in the evaluation of capital alternatives. Steps are as follows [Smith 1979]:

1. Identification of different alternatives.
2. Identification of different states of nature.

The outcomes (payoffs) for the various alternatives under different states of nature can now be calculated. Knowing the possible payoffs for each alternative, two further steps are required:

3. Assignment of a probability distribution to the payoffs under various states of nature for each alternative.
4. Aggregation of these payoffs into an expected value permitting identification of the optimal alternative.

There are two extreme cases in assigning probabilities to various states of nature. The first extreme case is that the decision maker has no information on which to base probability. Following the terminology suggested by Knight [1921], this case is referred as decision making under the context of uncertainty. One approach to handling problems in the context of uncertainty is the Bayes-Laplace criterion which assigns equal probabilities to various states of nature. Another approach is based on the extreme payoffs of these alternatives. This approach is generally referred to as the criteria of "maximin", "maximax", or "minimax regret".

The second and opposite extreme case is that the decision maker is able to determine, subjectively or objectively, the probabilities associated with the states of nature. Knight termed this case as decision making under the context of risk. In life-cycle cost analysis, it is difficult, if not impossible, to associate objective or subjective probabilities to a number of important variables, such as taxes, the forecasted escalation (or de-escalation) rate for different fuels, or the forecasted consumption of energy for a building. Hence, the traditional approach of maximizing the expected value is likely to be of little or no value to the decision maker.

In many practical decision problems, the assessment of probabilities lies somewhere between the two extreme cases. If the decision maker can rank the probabilities in the order of likelihood, the problem is referred as decision making under the context of incomplete knowledge. This term was coined by Cannon and Kmietowicz [1974].

Under the context of incomplete knowledge, the preordering of the probabilities may be done on the basis of either a weak or a strict ranking. Weak ranking only specifies the ordering of probabilities for various states of nature. Strict ranking assumes that successive probabilities differ from each other by at least a given amount.

For weak ranking under the context of incomplete knowledge, several decision criterion have been developed. Fishburn [1964] defined a strict dominance decision procedure. Cannon and Kmietowicz [1974] derived a partial average technique for determining the extreme expected values for an alternative. Kmietowicz and Pearman [1976] considered the dispersion (variance) of the payoffs under various states of nature as another criterion in choosing among alternatives. They also incorporated the expected value and variance into a single index with a trade-off coefficient (coefficient of risk aversion) between the expected value and the variance [Kmietowicz and Pearman 1981]. However, the method of determining the appropriate trade-off coefficient was not provided. Hence, the procedure for determining and final decision line when evaluating mutually exclusive alternatives is yet to be developed.

For strict ranking under the context of incomplete knowledge, Kmietowicz and Pearman [1981] extended the partial average technique to determine the extreme expected values for an alternative under a number of possible states of nature. However, the methodology of searching for the extreme variances and the extreme indexes of utility under the conditions of strict ranking has not been studied.

The overall objective of this research was to provide a methodology for determining a complete and final decision line when evaluating one set of mutually exclusive alternatives under the context of incomplete knowledge for both weak and strict ranking. Specific objectives were to:

1. Develop an algorithm to search for the extreme variances under conditions of strict ranking in the context of incomplete knowledge.
2. Develop an algorithm to search for the extreme index of utility which is a linear combination of the expected value and variance under conditions of strict ranking in the context of incomplete knowledge.
3. Improve the methodology to determine the appropriate value of the coefficient of risk aversion.
4. Determine the final decision line for mutually exclusive alternatives under conditions of weak ranking in the context of incomplete knowledge.
5. Determine the final decision line for mutually exclusive alternatives under conditions of strict ranking in the context of incomplete knowledge.

II. LITERATURE REVIEW

A decision model assumes that a decision maker can select one of (or rank) a number of strategies available to him. As the future is uncertain, the selected strategy must operate under one of a number of mutually exclusive states of nature. The actual payoff of the selected strategy, e.g., internal rate of return or present equivalent of savings, will depend on the state of nature which happens to occur.

The essential information of a decision problem may be conveniently summarized in a payoff matrix originating from von Neumann and Morgenstern [1947].

Strategy	State of Nature N_1	N_2	...	N_j	...	N_n
S_1	X_{11}	X_{12}	...	X_{1j}	...	X_{1n}
S_2	X_{21}	X_{22}	...	X_{2j}	...	X_{2n}
.	.	.		.		.
.	.	.		.		.
S_i	X_{i1}	X_{i2}	...	X_{ij}	...	X_{in}
.	.	.		.		.
.	.	.		.		.
S_m	X_{m1}	X_{m2}	...	X_{mj}	...	X_{mn}

Here, the symbols (N_1, N_2, ..., N_n) denote possible mutually exclusive states of nature. The decision maker knows that if he chooses strategy S_i and the environment dictates state of nature N_j, the result X_{ij} will be obtained. However, the decision maker does not know into which state of nature the future will fall.

The economic approach to a solution of this decision model is to search for an evaluation function for each strategy. The results from the evaluation function form the basis for final decision making.

There are three approaches to the classical decision model [Kmietowicz and Pearman 1981]. The three approaches are distinguished by the amount of information available about the probabilities with which the states of nature are likely to occur. The first approach assumes that there is no information about the probabilities available to the decision maker. This situation is referred to as decision making under uncertainty. The second approach assumes that the probabilities of the states of nature can be specified completely. This situation is referred to as decision making under risk. The third approach assumes that some information is available about the probabilities of the states of nature, but that the information is not comprehensive enough to enable exact specification of the probabilities. Decision making in such circumstances is referred to as decision making under conditions of incomplete knowledge.

A. Decision Making Under Uncertainty

Decision making under uncertainty assumes that the decision maker has no information about the probabilities of the states of nature. Several criteria have been proposed to help the decision maker face such conditions.

1. Maximin principle of Wald [1945] and of von Neumann and Morgenstern [1947]

The maximin criterion suggests that the decision maker should examine only the minimum payoffs and select the strategy with the largest of these. This criterion is very attractive to a cautious decision maker who wishes to ensure that even if an unfavorable state of nature occurs, there is a known minimum payoff below which he cannot fall.

2. Maximax principle of Keynes [1921]

The maximax criterion advises the decision maker to examine only maximum payoffs of strategies and to select the strategy with the largest of these maximum payoffs. The criterion reflects the viewpoint of a very optimistic decision maker who is greatly attracted by high payoffs and who hopes that the uncertain future develops favorably for him.

3. Optimism-pessimism index of Hurwicz [Milnor 1954]

The Hurwicz criterion suggests that the minimum and maximum payoffs of each strategy should be averaged using as weights α and $1 - \alpha$, where α is the index of pessimism. The strategy with the highest average is selected. The index α reflects the decision maker's attitude to risk taking. An extremely cautious decision maker will set $\alpha = 1$ and then the Hurwicz criterion reduces to the maximin criterion. An extremely aggressive decision maker will set $\alpha = 0$. Here the Hurwicz criterion reduces to the maximax criterion.

4. Minimax-regret principle of Savage [1951]

The minimax-regret criterion looks at the regret, opportunity cost or loss which arises when a particular state of nature is assumed to have occurred and the payoff of the selected strategy is smaller than the maximum payoff which could have been attained for that state of nature. The criterion takes the viewpoint of a cautious decision maker who wishes to ensure that the selected strategy does well in comparison with other strategies irrespective of which state of nature happens to arise.

5. Bayes-Laplace assumption of Bernouilli and Laplace [Sinn 1983]

The Bayes-Laplace criterion employs the principle of insufficient reason which postulates that if no information is available about the probabilities of the states of nature, it is only reasonable to assume that they are equally likely. The criterion thus reduces the problem to decision making under risk. The criterion goes on to suggest that the decision maker should calculate the expected payoff for each strategy and select the one with the highest payoff.

Except for the Bayes-Laplace criterion, most of the decision criteria completely ignore the intermediate payoffs. Kmietowicz and Pearman [1981] pointed out that a notable disadvantage of complete ignorance of the intermediate payoffs is their exclusive reliance on the extreme payoffs of each strategy. The intermediate payoffs may be more likely to occur than the extreme payoffs.

Another criticism of complete ignorance of the intermediate payoffs concerns the validity of the assumption of total lack of information about the probabilities of the states of nature. In many situations, the decision maker will be fairly confident that certain states of nature are more likely to occur than others.

B. Decision Making Under Risk

In decision making under risk, it is assumed that exact probabilities of the states of nature are available. On some occasions, the probabilities can be established experimentally or deduced from a _priori_ considerations; on other occasions, the decision maker's subjective probabilities are used.

1. Objective probability

The concept of objective probability is simply that value towards which the relative frequency of a particular event will stochastically converge when the decision situation is constantly repeated under indistinguishable conditions [Luce and Raiffa 1957]. The unambiguously correct objective probability does not exist. Also, it is impossible to repeatedly experiment under indistinguishable conditions in the decision making of the business world.

In the study of games of chance, a _priori_ judgment is usually made as to certain probability. The a _priori_ probabilities are the judgments of the relative uncertainty of various hypotheses made on the basis of all past information. The a _posteriori_ probabilities are the judgments made with the aid of new information. But for business affairs, there is no natural way of making these judgments. Instead, the appeal is to focus on past observations, if any [Arrow 1970].

2. Subjective probability

Subjective probabilities are based on the decision maker's beliefs about the future, and are obtained from him directly or indirectly in a number of ways. One attempt to deal with the difficulty of obtaining exact subjective probabilities of states of nature is attributable to Savage [1954]. Savage argued that, even if the decision maker is unable to specify exactly his subjective probabilities, deep down in his subconscious such a set must exist, and it is only necessary to elicit it from him. He suggested that this may be done by asking him a series of hypothetical questions about the probabilities of the states of nature. The replies can be analyzed with the help of laws governing probabilities, and an exact set of probabilities estimated.

Such experiments have, in fact, been conducted, but unfortunately it soon became apparent that the replies were often inconsistent. The greater the uncertainty about the future the greater was the number of inconsistent replies. These experiments seem to suggest that decision makers have some useful information about probabilities of states of nature, but it is not sufficiently detailed to ensure a unique specification. Other approaches for obtaining subjective probabilities are also described by Raiffa [1968], Hampton et al. [1973], Moore and Thomas [1975], and Sinn [1983].

3. Evaluation functions

Once the probabilities of the states of nature are established objectively or subjectively, it is possible to compare all possible alternatives by an evaluation function. There are four main types of evaluation functions that have been proposed: maximum expected value criterion, lexicographic criterion, expected utility criterion, and two-parametric substitutive criteria.

a. Maximum expected value criterion An important advantage of a maximum expected value criterion is the utilization of the probabilities of the states of nature and of all the payoffs of strategies. This approach to decision making does not exclude the possibility that another strategy may be preferable to the selected one under some states of nature. However, it does ensure that if many similar decisions are taken (payoffs and probabilities changing from problem to problem), the decision maker will do better in the long run than if he has employed the ignorance criterion [Kmietowicz and Pearman 1981].

A major criticism of the maximum expected value criterion is its unsuitability for unique and important decisions. Here the worst outcome of the selected strategy (if it occurs) may well ruin the decision maker financially, and it is no consolation for him to know that the strategy also contains a number of very attractive outcomes. Moreover, if a particular decision can be ruinous, there will be no possibility of offsetting the loss in the long run. Even if the loss can be sustained, it may take a long time to make it up.

Another criticism of the expected value criterion concerns the determination of the subjective probabilities of states of nature. It is argued that in many decision problems the decision maker is unable to specify the probabilities with any accuracy because of the uncertainty surrounding future events.

b. Lexicographic criterion Under this criterion, the decision maker is supposed to maximize the probabilities of the final payoffs exceeding some critical levels. The theory of lexicographic evaluation functions was developed by Roy [Sinn 1983], and extended by Encarnacion [1965].

c. Expected-utility criterion By means of a suitably chosen monotonically increasing index function, the payoff values are transformed into utilities. The mathematical expectation of these utility values serves as the evaluation function. The idea of expected utility criterion was developed by Bernouilli [1738]. The axiomatic foundation was developed by von Neumann and Morgenstern [1947].

d. Two-parametric substitutive criteria Two-parametric substitutive criteria are also called certainty equivalent criteria. From the probability distribution, two characteristic numbers are generated for each strategy to indicate the central tendency and dispersion of payoff values. The numbers are then evaluated by means of a substitutive evaluation function. Often, the evaluation function is illustrated graphically in a diagram by means of indifference curves. In addition to the indifference curves, the diagram contains an opportunity locus consisting of a number of points, each of which represents one possible strategy. The best strategy is then able to be identified.

Several two-parametric substitutive criteria have been developed using different statistical measures for the central tendency and dispersion. Lange [1943] used the mode and the range as the measures of central tendency and dispersion, respectively. Domar and Musgrave [1944] used the mean value as the measure of central tendency. As for the dispersion, they considered the expected value of all possible losses.

Fisher [1906] suggested an evaluation function that included the mean value and the standard deviation. Later, this approach was also discussed by Hicks [1933], Marschak [1938], Steindl [1941], Tintner [1941], and Lutz and Lutz [1951]. Thomas [1958] ranked the mean-standard deviation combinations of the alternatives according to a family of parallel, linear indifference curves to maximize the certainty equivalent, v:

$$v = \mu - A * \sigma$$

where A = a constant coefficient of risk aversion
μ = the mean value
σ = the standard deviation

Since, according to the mean-standard deviation evaluation function it does not matter whether changes in dispersion occur in the range of gain or loss, Markowitz [1970] suggested replacing the standard deviation by the semivariance which takes only the possible losses into account.

Constant [1983] applied the mean-standard deviation criteria to the context of uncertainty where a set of equal probabilities was assumed. By assuming that the coefficient of risk aversion is a product of a constant angular coefficient and the minimum attractive rate of return, Constant was able to determine the value of the angular coefficient. The final decision was then based on the incremental rates of return which is calculated from the following equation:

$$v = 0 = \overline{X} + a * m * \sigma$$

where v = the mean-standard deviation combination index
$\overline{X}$ = the mean value
a = the constant angular coefficient
m = the rates of return
σ = the standard deviation

C. Decision Making Under Incomplete Knowledge

In many practical decision problems, it is impossible to obtain estimates of the probabilities of future states of nature. However, the decision maker may have some information indicating that some states of nature are more likely to occur than others. This situation is referred to as decision making under incomplete knowledge.

Decision making models under incomplete knowledge can be split into two schools. The first school uses the step theory of probability or the expected probability to calculate an equivalent objective probability of each state of nature, thus reducing decision making under incomplete knowledge to a decision problem under risk. The step theory concept was first developed by Reichenbach [Sinn 1983] and later extended by Tintner [1941]. The basic idea of this theory is to transfer imprecise information, i.e., probabilities, into equivalent objective probabilities through a series of step transformations.

The expected probability distribution was developed by Agunwamba [1981]. Here, the uniform distribution is applied to the possible probability regions under the constraint of the probability of the hierarchies, thus obtaining the expected probability for each state of nature. Despite the fact that the probability distribution is an equivalent objective probability from the step function or an expected probability distribution, the problem still reduces to decision making under risk.

Instead of trying to establish an objective probability distribution from incomplete knowledge, an alternate approach was developed by Fishburn [1964]. By assuming that some knowledge of probabilities of states of nature is available in the form of a rank order of these probabilities (i.e., weak ranking of probabilities), Fishburn used this rank order of probabilities together with the payoff values of various states of nature to test whether the expected value of one particular strategy will always be greater than the expected value of an alternative strategy. In other words, Fishburn tested whether or not one strategy is statistically dominant over another (i.e., strict dominance).

By employing linear programming concepts, Cannon and Kmietowicz [1974] derived a partial average technique for determining the minimum and maximum expected values for any strategy under weak ranking of probabilities for various states of nature. Weak ranking of probabilities assumes the decision maker is only able to rank the probabilities of possible states of nature according to their likelihood to occur. Comparing these minimum and maximum expected values to the decision criteria under uncertainty, they showed that a decision maker who employs the extreme expected value method to guide his choice among strategies (rather than rely on pure maximum and minimum payoffs) makes better business decisions in the long run because the method takes full advantage of all the information available to the decision maker, including intermediate payoffs.

Kmietowicz and Pearman [1981] extended the partial average technique for determining the minimum and maximum expected value to a strict ranking of incomplete knowledge. The strict ranking of probabilities assumes that successive probabilities differ from each other by at least a given amount.

Since the expected values for each strategy are expressed in a range of the expected value, Kmietowicz and Pearman [1981] developed a new criterion of weak statistical dominance to extend the use of the maximum and minimum expected values. The weak statistical dominance is defined as follows:

> For two strategy S_1 and S_2, calculate the differences between the expected values of S_1 and S_2 for all possible corner boundary points. If $Max\{E(S_1)-E(S_2)\} > Max\{E(S_2)-E(S_1)\}$, then it can be said that strategy S_1 dominates S_2 weakly.

Kmietowicz and Pearman [1976] also considered a second parameter, the dispersion of the potential payoff values, as another criterion in choosing among alternatives. By using the variance as an index of dispersion, Kmietowicz and Pearman stated that only the corner point solutions needed to be considered in order to search for the maximum and minimum variance under weak ranking of probabilities. Kmietowicz and Pearman also discussed how the maximum variance might be used in practice.

Agunwamba [1980] pointed out an error by Kmietowicz and Pearman who ignored the existence of a special case. The special case was that solutions may exist inside the feasible region if there are only two distinct payoff values for n states of nature ($n \geq 3$). However, Agunwamba proved that there still exists a corner solution with maximum variance even if solutions exist inside the feasible region. Therefore, it is still correct to state that only the corner points needed to be considered in searching for the maximum and minimum variance under weak ranking of probabilities.

The search for the maximum and minimum variance under strict ranking was not studied by either Kmietowicz and Pearman, or Agunwamba.

Kmietowicz and Pearman [1981] also incorporated the expected value and variance into a single index with a trade-off coefficient between the expected value and variance under a weak ranking of probabilities. Kmietowicz and Pearman concluded that the assessment of the extreme values of such an index under weak ranking must proceed in the following way. If there are more than two distinct payoffs, only corner points can be optimal; if there are only two distinct payoff values, it is possible that a solution exists inside the feasible region. Hence, additional effort is required to search for the solution inside the feasible region which maximizes or minimizes the index of utility. However, depending on the relationship between the payoff values and the value of the coefficient of risk aversion, it need not always be the case that a solution inside the feasible region can exist for two distinct payoffs. Kmietowicz and Pearman did not extend their search for the extreme index to the case of strict ranking.

III. EXTREME VARIANCES OF PAYOFFS UNDER STRICT RANKING

One of the objectives in this research is to search for a set of probability values under strict ranking that cause the extreme values of the variance (maximum and minimum values). The objective function used to search for the extreme values is as follows:

$$VAR = \sum_{i=1}^{n} P_i * X_i^{\,2} - \left(\sum_{i=1}^{n} P_i * X_i \right)^2$$

In Section A, transformations take place that facilitate the search for the extreme values of the variance. In essence, the strict ranking constraints for the probabilities form the minimum requirements for the probabilities defined as M. Hence,

$$P_i = M_i + D_i$$

where P_i = Probability assigned to state of nature i
M_i = Minimum requirement of probability for state of nature i
D_i = Difference between the probability assigned and the minimum requirement for state of nature i

Subsequent transformations include T values which reflect the differences between the D values for each succeeding state of nature:

$$T_1 = D_1 - D_2$$

$$T_2 = D_2 - D_3$$

$$\vdots$$

$$T_n = D_n - D_{n+1} \quad (\text{where } D_{n+1} = 0)$$

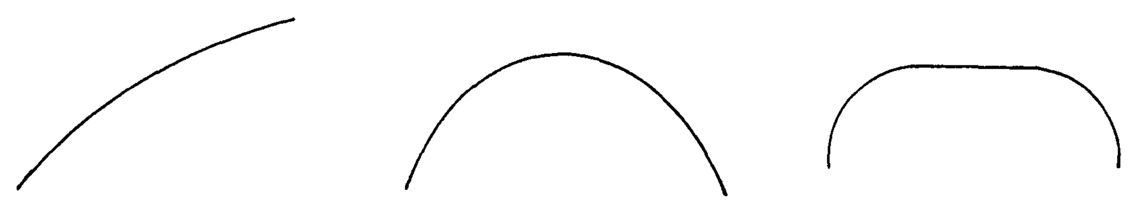
In order to keep the constraints of strict ranking, each D value must be greater than the succeeding D value. This can be done by limiting the T values to be not less than zero.

This transformation permits the function for the variance to be expressed in terms of T rather than P. It also simplifies the strict ranking constraints because all the requirements for P are now implied in T. The M terms are treated as constants in the objective function.

Section B examines the objective function with its constraints and recognizes it as a quadratic function subject to two linear constraints. The two linear constraints define the feasible region (combination of all feasible solutions) where the extreme values of the variance (maximum and minimum values) can lie.

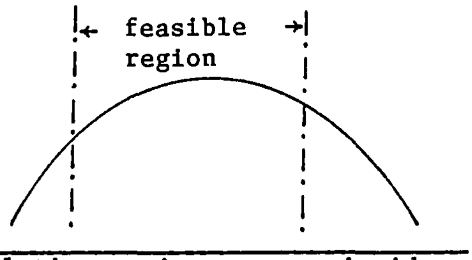
The negative quadratic term in the objective function dictates that the objective function is concave indicating that any relative extreme of the function must be a relative maximum. No relative minimum exists under any condition for the defined objective function. Therefore, the global minimum always occurs at one of the corner boundary points of the feasible region.

Section C describes the general approach that is used to search for the global maximum for the variance. Using a Lagrange multiplier and partial differentiation, a system of linear equations is formed. Any solution to the system of linear equations must be the relative maximum referred to in Section B. However, the system of equations may have no solution indicating that no relative maximum exists, one single solution (one relative maximum), or multiple solutions (multiple relative maximums) of equal value.

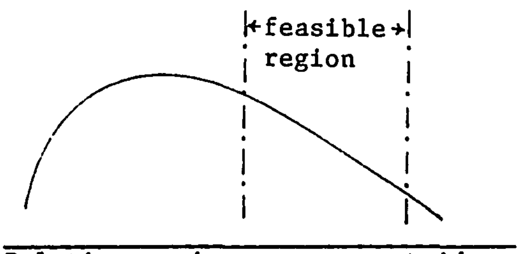


No solution   One solution   Multiple solutions

If a relative maximum formed by the concave function exists within the feasible region, it is also a global maximum. If no relative maximum exists, or the relative maximum(s) of the concave function lies outside of the feasible region, then the global maximum occurs at one of the corner boundary points of the feasible region. These corner boundary points are subsequently referred to as "corner solutions".



Relative maximum occurs inside of the feasible region. (relative maximum = global maximum)



Relative maximum occurs outside of the feasible region. (relative maximum $\neq$ global maximum)

In <u>Section D</u>, the objective function of the variance takes into consideration two possible states of nature. By applying the solving algorithm of the Lagrange multiplier and partial differentiation, it is possible to solve for the two T values of a single solution. Because one of the T values is negative, it can be concluded that the single relative maximum occurs outside of the feasible region. Hence, the global maximum and global minimum must occur at one of the corner boundary points of the feasible region.

In <u>Section E</u>, the objective function of the variance takes into consideration three possible states of nature. Using the same solving algorithm as in Section D, it is shown that the coefficient matrix for the system of equations is found to be zero. For a system of equations with zero determinant of the coefficient matrix, the system of equations has either no solution if the equations are contradictory to one another (insolvable), or multiple solutions if the equations are consistent (solvable).

It is shown that the equations are contradictory when all the three payoff values differ. This implies that a relative maximum does not exist. The global maximum must occur at the corner boundary points when there are three distinct payoff values.

The system of equations are consistent when one or two payoff values exist. This implies that multiple relative maximums exist. However, these multiple relative maximums may be located either inside or outside of the feasible region. If at least one relative maximum is located inside the feasible region, the relative maximum is the global maximum. If all the relative maximums are located outside the feasible region, the global maximum must occur at the corner boundary points.

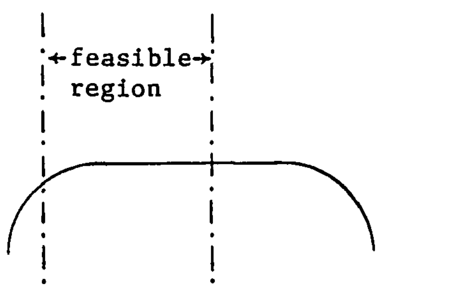
In Section F, the objective function of the variances takes into consideration more than two states of nature. Using the same solving algorithm, a system of n+1 equations is formed. The determinant of the coefficient matrix for the system of equations is found to be zero. Therefore, the system of equations has either no solution if the equations are contradictory to one another (insolvable conditions), or multiple solutions if the equations are consistent (solvable condition).

Solvable conditions for n states of nature are then dictated by Theorem I. For the system of equations to be solvable, Theorem I proves that there can be at most two distinct payoff values. Therefore, the global maximum must occur at the corner boundary points if there are more than two distinct payoff values. If there is only one payoff value, the variance is always equal to zero.

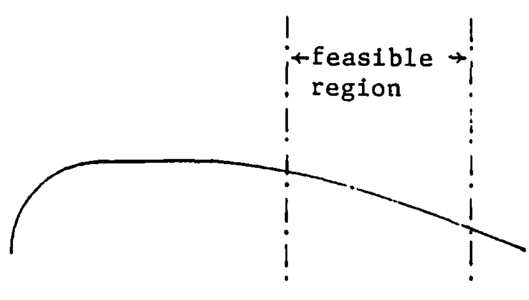
If there are two distinct payoff values, multiple relative maximums of equal value can exist. The necessary conditions for the existence of at least one relative maximum inside the feasible region are developed. If the necessary conditions are not met, the global maximum must occur at one of the corner boundary points.

If the necessary conditions are met, it is possible that a relative maximum(s) exists inside the feasible region. The multiple relative maximums can be located by a pair of linear equations. Any solution to the pair of linear equations results in a relative maximum. The multiple relative maximums have a common value of the variance which can be calculated directly. However, only the relative maximum(s) inside the feasible region defines the global maximum(s). In case that none of the multiple relative maximums is located inside the feasible region, the global maximum must occur at one of the corner boundary points.

Expressed as a two dimensional plane, the concave function has a flat top as shown in the following diagram.



At least one of the multiple relative maximums of equal value exists inside of the feasible region.



All multiple relative maximums of equal value exist outside of the feasible region.

A numerical example is provided to demonstrate the procedure for searching for the multiple relative maximum points.

A. Transformation of Objective Function and Constraints

The objective function used to search for the extreme variance under both weak and strict ranking is:

$$VAR = \sum_{i=1}^{n} P_i * [X_i - EXP]^2 = \sum_{i=1}^{n} P_i * X_i^2 - (\sum_{i=1}^{n} P_i * X_i)^2$$

where VAR = variance of payoff values under a set of probabilities
EXP = expected value of payoffs under a set of probabilities
P_i = probabilities assigned to state of nature i
X_i = payoff of state of nature i
n = number of possible states of nature

Setting the payoffs as constants under specified states of nature, the objective is to find the set of probabilities, $P = (P_1, P_2, \ldots, P_n)$, under which the extreme variances occur. The constraints of the probabilities under strict ranking are:

$$\sum_{i=1}^{n} P_i = 1$$

$P_i - P_{i+1} \geq k_i$ (for $i = 1, 2, \ldots, n-1, n$)

$P_i \geq 0$ (for $i = 1, 2, \ldots, n-1, n$)

$k_i \geq 0$ (for $i = 1, 2, \ldots, n-1, n$)

The minimum requirements of probabilities for state of nature i under strict ranking are symbolized as M_i:

$M_n = Min(P_n) = k_n$

$M_{n-1} = Min(P_{n-1}) = k_n + k_{n-1}$

.

$\cdot$

$\cdot$

$$M_{i+1} = \mathrm{Min}(P_{i+1}) = k_n + k_{n-1} + \dots + k_{i+1}$$

$$M_i = \mathrm{Min}(\ P_i\ ) = k_n + k_{n-1} + \dots + k_{i+1} + k_i$$

$\cdot$

$\cdot$

$\cdot$

$$M_3 = \mathrm{Min}(\ P_3\ ) = k_n + k_{n-1} + \dots + k_4 + k_3$$

$$M_2 = \mathrm{Min}(\ P_2\ ) = k_n + k_{n-1} + \dots + k_4 + k_3 + k_2$$

$$M_1 = \mathrm{Min}(\ P_1\ ) = k_n + k_{n-1} + \dots + k_4 + k_3 + k_2 + k_1$$

The summation of all the minimum requirements of probabilities can be symbolized as C_0:

$$C_0 = \sum_{i=1}^{n} M_i = (1 * k_1) + (2 * k_2) + \dots\dots + (n * k_n) = \sum_{i=1}^{n} i * k_i$$

Let D_i be denoted as the difference between the probability assigned to the state of nature i and its respective minimum requirement. Therefore,

$$P_i = M_i + D_i \quad (\text{for } i = 1, 2, \dots\dots, n-1, n)$$

Then the equation for calculating the variance can be rewritten as:

$$\mathrm{VAR} = \sum_{i=1}^{n} P_i * X_i^2 - [\sum_{i=1}^{n} P_i * X_i\ ]^2$$

$$= \sum_{i=1}^{n} (M_i + D_i) * X_i^2 - [\sum_{i=1}^{n} (M_i + D_i) * X_i\ ]^2$$

$$= \sum_{i=1}^{n} M_i\ X_i^2 + \sum_{i=1}^{n} D_i\ X_i^2 - [\sum_{i=1}^{n} M_i\ X_i + \sum_{i=1}^{n} D_i\ X_i\ ]^2$$

$$= \sum_{i=1}^{n} M_i\ X_i^2 + \sum_{i=1}^{n} D_i\ X_i^2 - [\sum_{i=1}^{n} M_i\ X_i\ ]^2 - [\sum_{i=1}^{n} D_i\ X_i\ ]^2$$

$$- 2 * [\sum_{i=1}^{n} M_i\ X_i\ ] * [\sum_{i=1}^{n} D_i\ X_i\ ]$$

Notice that the minimum requirements for the probabilities, M_i, are common to all possible probability combinations under strict ranking. The terms composed of M_i in the above equation are inactive in searching for the extreme variances. Hence, the terms composed of M_i and X_i can be defined as constants:

$$C_1 = \sum_{i=1}^{n} M_i X_i$$

$$C_2 = \sum_{i=1}^{n} M_i X_i^2$$

Then the equation to calculate the variance can be written as:

$$VAR = C_2 + \sum_{i=1}^{n} D_i X_i^2 - C_1^2 - [\sum_{i=1}^{n} D_i X_i]^2 - 2C_1 [\sum_{i=1}^{n} D_i X_i]$$

This equation can be greatly simplified by employing the transformations which were introduced by Kmietowicz and Pearman [1981]:

Let $T_i = D_i - D_{i+1}$ (for $i = 1, 2, \ldots, n$ and $D_{i+1} = 0$)

$$Y_i = \sum_{j=1}^{i} X_j \qquad \text{(for } i = 1, 2, \ldots, n)$$

$$Z_i = \sum_{j=1}^{i} X_j^2 \qquad \text{(for } i = 1, 2, \ldots, n)$$

Then,
$$\begin{aligned}\sum_{i=1}^{n} D_i X_i &= D_1 X_1 + D_2 X_2 + D_3 X_3 + \ldots + D_n X_n \\ &= [(D_1-D_2) + (D_2-D_3) + (D_3-D_4) + \ldots + (D_n-D_{n+1})]\ X_1 \\ &\quad + [(D_2-D_3) + (D_3-D_4) + \ldots + (D_n-D_{n+1})]\ X_2 \\ &\quad + [(D_3-D_4) + \ldots + (D_n-D_{n+1})]\ X_3 \\ &\quad \vdots \\ &\quad + [(D_n-D_{n+1})]\ X_n\end{aligned}$$

$$
\begin{aligned}
= & [T_1 + T_2 + T_3 + \ldots + T_n]\ X_1 \\
+ & \quad [T_2 + T_3 + \ldots + T_n]\ X_2 \\
+ & \qquad [T_3 + \ldots + T_n]\ X_3 \\
\cdot & \qquad\qquad\quad \cdot \\
\cdot & \qquad\qquad\quad \cdot \\
+ & \qquad\qquad\quad T_n\ X_n
\end{aligned}
$$

$$= T_1\ X_1 + T_2\ (X_1+X_2) + T_3\ (X_1+X_2+X_3) + \ldots\ldots + T_n \sum_{i=1}^{n} X_i$$

$$= T_1\ Y_1 + T_2\ Y_2 + T_3\ Y_3 + \ldots\ldots + T_n\ Y_n$$

$$= \sum_{i=1}^{n} T_i\ Y_i$$

Using the same transformation, the following results were also obtained.

$$\sum_{i=1}^{n} D_i\ X_i^{\,2} = \sum_{i=1}^{n} T_i\ Z_i$$

$$\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} i * T_i$$

The transformed objection function for calculating the variance is rewritten as:

$$VAR = C_2 + \sum_{i=1}^{n} T_i\ Z_i - C_1^{\,2} - [\sum_{i=1}^{n} T_i\ Y_i]^2 - 2C_1\ [\sum_{i=1}^{n} T_i\ Y_i] \qquad \text{(Eq. 3-1)}$$

As for the constraints:

Since, $\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} (M_i + D_i) = \sum_{i=1}^{n} M_i + \sum_{i=1}^{n} D_i = C_0 + \sum_{i=1}^{n} i * T_i = 1$

then, $$\sum_{i=1}^{n} i * T_i = 1 - C_0 \qquad \text{(Eq. 3-2)}$$

Because $P_i - P_{i+1} \geq k_i$ (for $i = 1, 2, \ldots, n$)

$$(M_i+D_i) - (M_{i+1}+D_{i+1}) \geq k_i$$

$$(M_i-M_{i+1}) + (D_i-D_{i+1}) \geq k_i$$

$$[(k_n + \cdots + k_{i+1} + k_i) - (k_n + \cdots + k_{i+1})] + (D_i - D_{i+1}) \geq k_i$$

$$k_i + (D_i - D_{i+1}) \geq k_i$$

then, $D_i - D_{i+1} \geq 0$

i.e., $T_i \geq 0$ (for $i = 1, 2, \ldots, n$) (Eq. 3-3)

The transformed objective function and constraints in the search for the extreme variances under strict ranking are summarized as functions of T, where T is a n-component vector $(T_1, T_2, \ldots, T_n)$.

Maximizing or minimizing

$$VAR = C_2 + \sum_{i=1}^{n} T_i Z_i - C_1^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1 [\sum_{i=1}^{n} T_i Y_i] \quad \text{(Eq. 3-1)}$$

Subject to

$$\sum_{i=1}^{n} i * T_i = 1 - C_0 \quad \text{(Eq. 3-2)}$$

$$T_i \geq 0 \quad (\text{for } i = 1, 2, 3, \ldots, n) \quad \text{(Eq. 3-3)}$$

Notice that all of the k_i's are set equal to zero under weak ranking. Hence, the minimum requirement of probabilities, M_i, for all states of nature are equal to zero. As a result, the constants (C_0, C_1, and C_2) which are composed of M_i are equal to zero. Therefore, Eq. 3-1 can be rewritten as the following equation for weak ranking.

$$VAR = \sum_{i=1}^{n} T_i Z_i - [\sum_{i=1}^{n} T_i Y_i]^2$$

Under weak ranking, Eq. 3-2 becomes,

$$\sum_{i=1}^{n} i * T_i = 1$$

The above two equations are the same as the objective function and constraint derived by Kmietowicz and Pearman [1976]. Therefore, searching for the extreme variances under weak ranking is a special case of the problem under strict ranking where all of the k_i's ($i = 1, 2, .., n$) are set equal to zero.

B. Nature of Transformed Objective Function and Constraints

Since the objective function is a quadratic function and the constraints are linear equations, the problem to be solved is in the form of quadratic programming. The objective function is formed by one quadratic term, $-[\sum_{i=1}^{n} T_i Y_i]^2$, together with linear terms and constants. Since this quadratic form is always less than or equal to zero, it is defined as negative semidefinite. Any negative semidefinite quadratic form is a concave function over all of Euclidean n-dimensional space, E^n [Theorem 3.4, Simmons 1975]. In addition, the linear terms are both convex and concave over all of E^n [Theorem 3.1, Simmons 1975]. The sum of two or more concave functions is concave [Theorem 3.3, Simmons 1975]. Therefore, the objective function, which is the sum of one concave quadratic term and several linear terms, is a concave function over all of E^n. As a result, a relative extreme point must be a relative maximum if such a relative extreme point exists.

A feasible solution is a solution for which all the constraints are satisfied. A feasible region is a collection of all feasible solutions. The equality constraint, Eq. 3-2, defines the feasible region as a hyperplane in E^n. A hyperplane in n-space is a convex set and is analogous to the line in two-dimensional space and the plane in three-dimensional space. The n non-negativity constraints, Eq. 3-3, confine the feasible region into n closed positive half-spaces (which are convex sets as well). The feasible region is then the intersection of the hyperplane and the n closed positive half-spaces. Since the intersection of any two convex sets must itself be convex, these constraints together define the feasible region for this problem as a closed convex set.

C. General Approach to Search for Extreme Variances

Since the objective function of the variance is concave over all of E^n, the global maximum of the variance within the feasible region may either be determined by the relative maximum(s) if it exists inside the feasible region, or be located at boundary points of the feasible region. The global minimum of the variance within the feasible region must occur at the boundary points because no relative minimum exists.

In searching for the global extremes (maximum and minimum), it is interesting to examine the characteristics of the boundary points of the feasible region. For an n-dimensional problem, there are n boundaries that form the feasible region because the feasible region is a hyperplane confined by n half-spaces. Each boundary consists of feasible solutions which lie on the intersection of the hyperplane and at least one half-space. Since the hyperplane and the half-spaces are all closed convex sets, the boundary itself is a closed convex set. Considering one single boundary as if it were the only feasible region, at least one global extreme occurs at the "boundary points" of this boundary [Theorem 3.8, Simmons 1975]. The reasoning can be repeated until the corner boundary points of the feasible region are reached. Therefore, consideration of all boundary points has been reduced to only the corner boundary points in searching for the global extremes.

A corner boundary point lies at the intersection of n constraint boundaries. Because of the equality constraint Eq. 3-2 and the n non-negativity constraints Eq. 3-3, a total of n+1 corner boundary points are possible. However, the intersection point of n non-negativity constraint boundaries is located outside the feasible region since it violates Eq. 3-2. Therefore, there are n corner boundary points for the feasible region. The n corner boundary points are characterized by having only one T_i value that satisfies Eq. 3-2, keeping all the other n-1 T_i values at zero. Expressed in equations, the n corner boundary points for the feasible region are:

$$T_i = (1-C_0)/i \qquad \text{(for } i = 1, 2, \ldots, \text{ or } n)$$
$$T_j = 0 \qquad \text{(for } j = 1, 2, \ldots, n; \text{ but } j \neq i)$$

To locate the relative maximum for the constrained quadratic programming problem, the method of Lagrange multipliers can be used. By introducing a Lagrange multiplier, the objective function is combined with the constraint (not including the non-negativity constraints) into a Lagrange function.

Setting the first partial derivatives of the Lagrange function equal to zero, a system of n+1 linear equations of n variables (T_i's) and the Lagrange multiplier is generated. A solution that satisfies the system of n+1 equations is a relative maximum of the original quadratic programming problem. Since there are n+1 equations for n+1 unknowns (T_i's and the Lagrange multiplier), only one relative maximum exists in E^n if the n+1 linear equations are independent of one another. However, if some of the n+1 equations are dependent on one another (in which case the determinant of the coefficient matrix will be equal to zero), either no relative maximum or multiple relative maximums exists in E^n. Since the Lagrange function ignores the non-negativity constraints, the relative maximum(s) may be located outside the feasible region.

Therefore, there are three possible situations in searching for the global maximum:

a) If at least one relative maximum exists within the feasible region, the constrained relative maximum(s) must be a global maximum(s) over the feasible region [Theorem 3.7, Simmons 1975].

b) If a relative maximum(s) exists, but the relative maximum(s) is located outside the feasible region, then the global maximums must occur at the corner boundary points.

c) If there is no relative maximum, then the global maximum must occur at the corner boundary points.

D. Extreme Variances for Two States of Nature

The objective function and its constraints for two possible states of nature are:

Maximizing or minimizing

$$VAR = C_2 + T_1Z_1 + T_2Z_2 - C_1{}^2 - [T_1Y_1+T_2Y_2]^2 - 2C_1\ (T_1Y_1+T_2Y_2)$$

Subject to

$$T_1 + 2T_2 = 1 - C_0$$

$$T_1 \geq 0,\ T_2 \geq 0$$

Since the objective function is concave, the global minimum must occur at the corner boundary points. In order to determine the global maximum, it is necessary to ascertain whether or not a relative maximum exists inside the feasible region. Temporarily ignoring the non-negativity constraints, a standard Lagrange function can be formed to search for the relative maximum variance. The Lagrange function is:

$$\begin{aligned} L &= C_2 + T_1Z_1 + T_2Z_2 - C_1{}^2 - [T_1Y_1 + T_2Y_2]^2 - 2C_1\ (T_1Y_1 + T_2Y_2) \\ &\quad + \lambda * [1 - C_0 - T_1 - 2T_2] \\ &= C_2 + T_1Z_1 + T_2Z_2 - C_1{}^2 - T_1{}^2Y_1{}^2 - T_2{}^2Y_2{}^2 - 2T_1T_2Y_1Y_2 - 2C_1T_1Y_1 \\ &\quad - 2C_1T_2Y_2 + \lambda[1 - C_0 - T_1 - 2T_2] \end{aligned}$$

For a relative maximum to exist, it is necessary that the system of three simultaneous equations obtained from the partial derivatives of the Lagrange function with respect to T_1, T_2, and λ be solvable [Schmidt 1974, p. 326].

$$\frac{\partial L}{\partial T_1} = Z_1 - 2T_1Y_1{}^2 - 2T_2Y_1Y_2 - 2C_1Y_1 - \lambda = 0 \qquad \text{(Eq. 3-4)}$$

$$\frac{\partial L}{\partial T_2} = Z_2 - 2T_2Y_2{}^2 - 2T_1Y_1Y_2 - 2C_1Y_2 - 2\lambda = 0 \qquad \text{(Eq. 3-5)}$$

$$\frac{\partial L}{\partial \lambda} = 1 - C_0 - T_1 - 2T_2 = 0 \qquad \text{(Eq. 3-6)}$$

Multiplied by (Y_2/Y_1), Eq. 3-4 becomes:

$$Z_1(Y_2/Y_1) - 2T_1Y_1Y_2 - 2T_2Y_2^2 - 2C_1Y_2 - \lambda(Y_2/Y_1) = 0$$

Subtracting from Eq. 3-5,

$$Z_2 - Z_1(Y_2/Y_1) - \lambda(2 - Y_2/Y_1) = 0$$

Multiplying by Y_1,

$$Z_2Y_1 - Z_1Y_2 - \lambda(2Y_1 - Y_2) = 0$$

Therefore, the value for λ is obtained:

$$\lambda = (Z_2Y_1 - Z_1Y_2) / (2Y_1 - Y_2) \qquad \text{(Eq. 3-7)}$$

Substitute the expressions for T_1 and λ (Eq. 3-6 and Eq. 3-7) into Eq. 3-5, and solve for T_2.

$$Z_2 - 2T_2Y_2^2 - 2(1-C_0-2T_2)Y_1Y_2 - 2C_1Y_2 - 2(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2) = 0$$

$$Z_2 - 2T_2Y_2^2 - 2(1-C_0)Y_1Y_2 + 4T_2Y_1Y_2 - 2C_1Y_2 - 2(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2)=0$$

$$T_2(2Y_2^2-4Y_1Y_2) = Z_2 - 2(1-C_0)Y_1Y_2 - 2C_1Y_2 - 2(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2)$$

Substitute $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Z_1 = X_1^2$, and $Z_2 = X_1^2 + X_2^2$,

$$T_2[2(X_1+X_2)^2-4X_1(X_1+X_2)] = (X_1^2+X_2^2) - 2(1-C_0)X_1(X_1+X_2) - 2C_1(X_1+X_2)$$
$$- 2[(X_1^2+X_2^2)X_1-X_1^2(X_1+X_2)]/[2X_1-(X_1+X_2)]$$

$$T_2[2X_1^2+2X_2^2+4X_1X_2-4X_1^2-4X_1X_2] = X_1^2 + X_2^2 - 2X_1^2 - 2X_1X_2$$
$$+ 2C_0X_1(X_1+X_2) - 2C_1(X_1+X_2)$$
$$- 2[X_1^3+X_1X_2^2-X_1^3-X_1^2X_2]/[2X_1-X_1-X_2]$$

$$T_2[2X_2^2-2X_1^2] = X_2^2 - X_1^2 - 2X_1X_2 + 2C_0X_1(X_1+X_2) - 2C_1(X_1+X_2)$$
$$- 2[X_1X_2^2-X_1^2X_2]/[X_1-X_2]$$

$$2T_2[X_2^2-X_1^2] = X_2^2 - X_1^2 - 2X_1X_2 + 2C_0X_1(X_1+X_2) - 2C_1(X_1+X_2) + 2X_1X_2$$

$$2T_2[X_2^2-X_1^2] = (X_2^2-X_1^2) + 2C_0X_1(X_1+X_2) - 2C_1(X_1+X_2)$$

$$T_2 = 1/2 + 2C_0X_1(X_1+X_2)/2(X_2^2-X_1^2) - 2C_1(X_1+X_2)/2(X_2^2-X_1^2)$$

$$T_2 = 1/2 + C_0X_1/(X_2-X_1) - C_1/(X_2-X_1)$$

Recall the definition for C_0 and C_1,

$$C_0 = 1k_1 + 2k_2$$

$$C_1 = M_1X_1 + M_2X_2 = (k_1+k_2)X_1 + k_2X_2$$

Substituting C_0 and C_1, the expression for T_2 can be further simplified.

$$\begin{aligned} T_2 &= 1/2 + (k_1+2k_2)X_1/(X_2-X_1) - [(k_1+k_2)X_1+k_2X_2]/(X_2-X_1) \\ &= 1/2 + [k_1X_1 + 2k_2X_1 - k_1X_1 - k_2X_1 - k_2X_2] / (X_2-X_1) \\ &= 1/2 + [k_2X_1 - k_2X_2] / (X_2-X_1) \\ &= 1/2 - [k_2(X_2-X_1)] / (X_2-X_1) \\ &= 1/2 - k_2 \end{aligned}$$

Since $T_1 + 2T_2 = 1 - C_0$,

$$\begin{aligned} T_1 &= 1 - C_0 - 2T_2 \\ &= 1 - k_1 - 2k_2 - 1 + 2k_2 \\ &= -k_1 \end{aligned}$$

A single solution ($T_1 = -k_1$, $T_2 = 1/2- k_2$) results from the system of three equations causing a single relative maximum point. Because $T_1 = -k_1$ and $k_1 \geq 0$, T_1 is always less than or equal to zero which is contrary to the non-negativity constraint of T_1 (Eq. 3-3). Hence, the single relative maximum must be located outside the feasible region. As a result, both the global maximum and minimum variance for two states of nature always occurs at the corner boundary points.

E. Extreme Variances for Three States of Nature

Before extending the case of two states of nature to the case of n states of nature, it is interesting to examine the case of three states of nature because it represents the simplest form of the n-dimensional case.

The objective function and its constraints for three states of nature are:

Maximizing or minimizing

$$VAR = C_2+T_1Z_1+T_2Z_2+T_3Z_3-C_1{}^2-[T_1Y_1+T_2Y_2+T_3Y_3]^2-2C_1(T_1Y_1+T_2Y_2+T_3Y_3)$$

Subject to

$$T_1 + 2T_2 + 3T_3 = 1 - C_0$$

$$T_1 \geq 0, \ T_2 \geq 0, \ T_3 \geq 0$$

Since the objective function is concave over all of E^3, the global maximum variance within the feasible region may either be determined by the relative maximum(s) if it exists inside the feasible region, or be located at boundary points of the feasible region. The global minimum variance within the feasible region must occur at the boundary points because no relative minimum exists.

1. Lagrange function

The method of Lagrange multiplier is used to search for the relative maximum(s). Temporarily ignoring the non-negativity constraints, a standard Lagrange function can be formed.

$$\begin{aligned} L &= C_2+T_1Z_1+T_2Z_2+T_3Z_3-C_1{}^2-[T_1Y_1+T_2Y_2+T_3Y_3]^2-2C_1(T_1Y_1+T_2Y_2+T_3Y_3) \\ &\quad + \lambda[1-C_0-T_1-2T_2-3T_3] \\ &= C_2+T_1Z_1+T_2Z_2+T_3Z_3-C_1{}^2-T_1{}^2Y_1{}^2-T_2{}^2Y_2{}^2-T_3{}^2Y_3{}^2-2T_1T_2Y_1Y_2-2T_1T_3Y_1Y_3 \\ &\quad -2T_2T_3Y_2Y_3-2C_1T_1Y_1-2C_1T_2Y_2-2C_1T_3Y_3 + \lambda[1-C_0-T_1-2T_2-3T_3] \end{aligned}$$

For a relative maximum to exist, it is necessary that the four simultaneous equations obtained from the partial derivatives of the Lagrange function with respect to T_1, T_2, T_3, and λ be solvable [Schmidt 1974].

$$\frac{\partial L}{\partial T_1} = Z_1 - 2T_1Y_1^2 - 2T_2Y_1Y_2 - 2T_3Y_1Y_3 - 2C_1Y_1 - \lambda = 0 \qquad \text{(Eq. 3-8)}$$

$$\frac{\partial L}{\partial T_2} = Z_2 - 2T_2Y_2^2 - 2T_1Y_1Y_2 - 2T_3Y_2Y_3 - 2C_1Y_2 - 2\lambda = 0 \qquad \text{(Eq. 3-9)}$$

$$\frac{\partial L}{\partial T_3} = Z_3 - 2T_3Y_3^2 - 2T_1Y_1Y_3 - 2T_2Y_2Y_3 - 2C_1Y_3 - 3\lambda = 0 \qquad \text{(Eq. 3-10)}$$

$$\frac{\partial L}{\partial \lambda} = 1 - C_0 - T_1 - 2T_2 - 3T_3 = 0 \qquad \text{(Eq. 3-11)}$$

Eqs. 3-8, 3-9, 3-10, and 3-11 can be written in the form of matrixes:

$$\begin{bmatrix} \frac{\partial L}{\partial T_1} \\ \frac{\partial L}{\partial T_2} \\ \frac{\partial L}{\partial T_3} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2Y_1^2 & 2Y_1Y_2 & 2Y_1Y_3 & 1 \\ 2Y_1Y_2 & 2Y_2^2 & 2Y_2Y_3 & 2 \\ 2Y_1Y_3 & 2Y_2Y_3 & 2Y_3^2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \lambda \end{bmatrix} - \begin{bmatrix} 2C_1Y_1 \\ 2C_1Y_2 \\ 2C_1Y_3 \\ C_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rearrange the matrix equation,

$$\begin{bmatrix} 2Y_1^2 & 2Y_1Y_2 & 2Y_1Y_3 & 1 \\ 2Y_1Y_2 & 2Y_2^2 & 2Y_2Y_3 & 2 \\ 2Y_1Y_3 & 2Y_2Y_3 & 2Y_3^2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} Z_1-2C_1Y_1 \\ Z_2-2C_1Y_2 \\ Z_3-2C_1Y_3 \\ 1-C_0 \end{bmatrix}$$

Divide the first three rows by $2Y_1$, $2Y_2$, and $2Y_3$, respectively,

$$\begin{bmatrix} Y_1 & Y_2 & Y_3 & 1/2Y_1 \\ Y_1 & Y_2 & Y_3 & 2/2Y_2 \\ Y_1 & Y_2 & Y_3 & 3/2Y_3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} (Z_1/2Y_1)-C_1 \\ (Z_2/2Y_2)-C_1 \\ (Z_3/2Y_3)-C_1 \\ 1-C_0 \end{bmatrix}$$

Subtract rows 2 and 3 by row 1, respectively,

$$\begin{bmatrix} Y_1 & Y_2 & Y_3 & 1/2Y_1 \\ 0 & 0 & 0 & (2/2Y_2)-(1/2Y_1) \\ 0 & 0 & 0 & (3/2Y_3)-(1/2Y_1) \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} (Z_1/2Y_1)-C_1 \\ (Z_2/2Y_2)-(Z_1/2Y_1) \\ (Z_3/2Y_3)-(Z_1/2Y_1) \\ 1-C_0 \end{bmatrix}$$

The determinant of the four by four coefficient matrix is calculated as:

$$-(1/2Y_1)\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} + [(2/2Y_2)-(1/2Y_1)]\begin{vmatrix} Y_1 & Y_2 & Y_3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$-\ [(3/2Y_3)-(1/2Y_1)]\begin{vmatrix} Y_1 & Y_2 & Y_3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} + 0\begin{vmatrix} Y_1 & Y_2 & Y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= -(1/2Y_1)(0) + [(2/2Y_2)-(1/2Y_1)](0) - [(3/2Y_3)-(1/2Y_1)](0) + 0(0)$$

$$= 0$$

The singular coefficient matrix implies that the system of four equations as a result of the partial derivatives has either no solution or multiple solutions. The system of equations has no solution when these equations are contradictory to one another (insolvable conditions). On the other hand, if these equations are consistent (solvable conditions), the system of equations has multiple solutions.

For example, the following system of equations is said to be solvable because numerous sets of X values can be found that can be simultaneously applied to both equations:

$$X_1 + 2X_2 = 3$$

$$2X_1 + 4X_2 = 6$$

Furthermore, a system of equations is said to be insolvable because no set of X values can be found that satisfy both equations:

$$X_1 + 2X_2 = 3$$

$$2X_1 + 4X_2 = 11$$

2. Solvable conditions for three states of nature

It is important to recognize the conditions for which the system of equations has no solution (insolvable conditions) or multiple solutions (solvable conditions). If the system of equations is insolvable, no relative maximum exists. If the system of equation is solvable, there are multiple relative maximums of equal value.

Eqs. 3-8, 3-9, and 3-10 can be simplified by dividing the three equations by Y_1, Y_2, and Y_3, respectively. The results are:

$$Z_1/Y_1 - 2T_1Y_1 - 2T_2Y_2 - 2T_3Y_3 - 2C_1 - \lambda/Y_1 = 0 \qquad \text{(Eq. 3-12)}$$

$$Z_2/Y_2 - 2T_1Y_1 - 2T_2Y_2 - 2T_3Y_3 - 2C_1 - 2\lambda/Y_2 = 0 \qquad \text{(Eq. 3-13)}$$

$$Z_3/Y_3 - 2T_1Y_1 - 2T_2Y_2 - 2T_3Y_3 - 2C_1 - 3\lambda/Y_3 = 0 \qquad \text{(Eq. 3-14)}$$

From the three equations, it is observed that the coefficients for

T_1, T_2, and T_3 are all identical, and only the constants and the coefficients for λ are different. Therefore, three results of λ can be derived from Eqs. 3-12, 3-13, and 3-14 (one result from each pair of equations). In order to determine if these three equations are consistent, it is necessary to examine the three resulting values of λ. If these three values of λ are identical, the three equations (Eqs. 3-12, 3-13, and 3-14; or equivalently Eqs. 3-8, 3-9, and 3-10) are consistent. And the system of three equations is solvable as a result. If these three values of λ are not identical, the three equations are contradictory to one another. And the system of equations is insolvable.

The first result, λ_1, can be derived by subtracting Eq. 3-12 from Eq. 3-13.

$$Z_2/Y_2 - Z_1/Y_1 - 2\lambda/Y_2 + \lambda/Y_1 = 0$$

$$\begin{aligned}\lambda_1 &= [Z_2Y_1-Z_1Y_2] \;/\; [2Y_1-Y_2] \\ &= [(X_1{}^2+X_2{}^2)(X_1)-(X_1{}^2)(X_1+X_2)] \;/\; [2(X_1)-(X_1+X_2)] \\ &= [X_1X_2(X_2-X_1)] \;/\; [X_1-X_2] \\ &= -\,X_1X_2\end{aligned}$$

The second result, λ_2, can be derived by subtracting Eq. 3-12 from Eq. 3-14.

$$Z_3/Y_3 - Z_1/Y_1 - 3\lambda/Y_3 + \lambda/Y_1 = 0$$

$$\begin{aligned}\lambda_2 &= [Z_3Y_1-Z_1Y_3] \;/\; [3Y_1-Y_3] \\ &= [(X_1{}^2+X_2{}^2+X_3{}^2)(X_1)-(X_1{}^2)(X_1+X_2+X_3)] \;/\; [3(X_1)-(X_1+X_2+X_3)] \\ &= [X_1X_2(X_2-X_1)+X_1X_3(X_3-X_1)] \;/\; [(X_1-X_2)+(X_1-X_3)]\end{aligned}$$

The third result, λ_3, can be derived by subtracting Eq. 3-13 from Eq. 3-14.

$$Z_3/Y_3 - Z_2/Y_2 - 3\lambda/Y_3 + 2\lambda/Y_2 = 0$$

$$\lambda_3 = [Z_3Y_2-Z_2Y_3] \ / \ [3Y_2-2Y_3]$$

$$= \frac{[(X_1{}^2+X_2{}^2+X_3{}^2)(X_1+X_2)-(X_1{}^2+X_2{}^2)(X_1+X_2+X_3)]}{[3(X_1+X_2)-2(X_1+X_2+X_3)]}$$

$$= [X_1X_3(X_3-X_1)+X_2X_3(X_3-X_2)] \ / \ [(X_1-X_3)+(X_2-X_3)]$$

Examining the expressions for the three results, it is observed that 1) the three results are not identical in value if X_1, X_2, and X_3 are all different from one another; 2) the three results will be identical in value only for any one of the following four special conditions:

a. $X_1 = X_2 = X_3$

b. $X_1 = X_2$, and $X_1 \neq X_3$

c. $X_1 = X_3$, and $X_1 \neq X_2$

d. $X_2 = X_3$, and $X_1 \neq X_2$

Notice that these four special conditions could also be described as having only one or two distinct payoff values. Therefore, Eqs. 3-8, 3-9, and 3-10 are consistent if there are only one or two distinct payoff values. In other words, the system of equations is solvable if there are only one or two distinct payoff values for three states of nature. On the other hand, Eqs. 3-8, 3-9, and 3-10 are contradictory if there are three distinct payoff values. In other words, the system of equations is insolvable if there are more than two distinct payoff values.

3. Summary of solutions for three states of nature

For three states of nature, the global minimum variance always occurs at the corner boundary points. If all three payoff values are different, the global maximum variance will also occur at the corner boundary points since no relative maximum exists. If there are only 1 or 2 distinct payoff values, multiple relative maximums exist. The global maximum will be determined by the relative maximum which is located inside the feasible region. The global maximum will still occur at the corner boundary points if all the relative maximums are located outside the feasible region.

F. Extreme Variances for N States of Nature

The objective function and constraints in the search for the extreme variances for n states of nature (where $n \geq 3$) under strict ranking are:

Maximizing or minimizing

$$VAR = C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1 [\sum_{i=1}^{n} T_i Y_i] \qquad \text{(Eq. 3-1)}$$

Subject to

$$\sum_{i=1}^{n} i * T_i = 1 - C_0 \qquad \text{(Eq. 3-2)}$$

$$T_i \geq 0 \quad \text{(for } i = 1, 2, 3, \ldots, n) \qquad \text{(Eq. 3-3)}$$

Since the objective function is concave, no relative minimum exists. The global minimum must occur at the corner boundary points. In order to determine the global maximum, it is necessary to ascertain whether or not a relative maximum exists inside the feasible region. A Lagrange function is used to search for the relative maximum(s).

1. Lagrange function

The appropriate Lagrange function for n states of nature is:

$$L = C_2 + \sum_{i=1}^{n} T_i Z_i - C_1^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1[\sum_{i=1}^{n} T_i Y_i] + \lambda[1-C_0-\sum_{i=1}^{n} iT_i]$$

For a relative maximum to exist, it is necessary that the system of n+1 simultaneous linear equations obtained from the partial derivatives of the Lagrange function with respect to the T_i values (i = 1, 2, 3, ... , n) and to λ, be solvable for T_i and λ [Schmidt 1974]. That is, it must be possible to solve the following n+1 equations simultaneously:

$$\frac{\partial L}{\partial T_i} = Z_i - 2Y_i \sum_{j=1}^{n} T_j Y_j - 2C_1 Y_i - i\lambda = 0 \quad (i = 1, 2, .., n) \qquad \text{(Eq. 3-15)}$$

$$\frac{\partial L}{\partial \lambda} = 1 - C_0 - \sum_{i=1}^{n} iT_i = 0 \quad \text{or} \quad \sum_{i=1}^{n} iT_i = 1 - C_0 \qquad \text{(Eq. 3-2)}$$

Eqs. 3-15 and 3-2 can be written in the form of matrixes:

$$\begin{bmatrix} \frac{\partial L}{\partial T_1} \\ \frac{\partial L}{\partial T_2} \\ \vdots \\ \frac{\partial L}{\partial T_n} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \\ 1 \end{bmatrix} - \begin{bmatrix} 2Y_1^2 & 2Y_1Y_2 & \cdots & 2Y_1Y_n & 1 \\ 2Y_1Y_2 & 2Y_2^2 & \cdots & 2Y_2Y_n & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 2Y_1Y_n & 2Y_2Y_n & \cdots & 2Y_n^2 & n \\ 1 & 2 & \cdots & n & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \\ \lambda \end{bmatrix} - \begin{bmatrix} 2C_1Y_1 \\ 2C_1Y_2 \\ \vdots \\ 2C_1Y_n \\ C_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Rearrange the matrix equation,

$$
\begin{bmatrix}
2Y_1^2 & 2Y_1Y_2 & \cdots & 2Y_1Y_n & 1 \\
2Y_1Y_2 & 2Y_2^2 & \cdots & 2Y_2Y_n & 2 \\
\vdots & \vdots & & \vdots & \vdots \\
2Y_1Y_n & 2Y_2Y_n & \cdots & 2Y_n^2 & n \\
1 & 2 & \cdots & n & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\ T_2 \\ \vdots \\ T_n \\ \lambda
\end{bmatrix}
=
\begin{bmatrix}
Z_1-2C_1Y_1 \\ Z_2-2C_1Y_2 \\ \vdots \\ Z_n-2C_1Y_n \\ 1-C_0
\end{bmatrix}
$$

Divide the first n rows by $2Y_1$, $2Y_2$, ..., and $2Y_n$, respectively,

$$
\begin{bmatrix}
Y_1 & Y_2 & \cdots & Y_n & 1/2Y_1 \\
Y_1 & Y_2 & \cdots & Y_n & 2/2Y_2 \\
\vdots & \vdots & & \vdots & \vdots \\
Y_1 & Y_2 & \cdots & Y_n & 3/2Y_n \\
1 & 2 & \cdots & n & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\ T_2 \\ \vdots \\ T_n \\ \lambda
\end{bmatrix}
=
\begin{bmatrix}
(Z_1/2Y_1)-C_1 \\ (Z_2/2Y_2)-C_1 \\ \vdots \\ (Z_n/2Y_n)-C_1 \\ 1-C_0
\end{bmatrix}
$$

Subtract rows 2, 3, ..., n by row 1, respectively,

$$
\begin{bmatrix}
Y_1 & Y_2 & \cdots & Y_n & 1/2Y_1 \\
0 & 0 & \cdots & 0 & (2/2Y_2)-(1/2Y_1) \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & (n/2Y_n)-(1/2Y_1) \\
1 & 2 & \cdots & n & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\ T_2 \\ \vdots \\ T_n \\ \lambda
\end{bmatrix}
=
\begin{bmatrix}
(Z_1/2Y_1)-C_1 \\ (Z_2/2Y_2)-(Z_1/2Y_1) \\ \vdots \\ (Z_n/2Y_n)-(Z_1/2Y_1) \\ 1-C_0
\end{bmatrix}
$$

The determinant of the (n+1) by (n+1) coefficient matrix must be equal to zero [Cofactor method of calculating the determinant, Schmidt 1974]. The singular coefficient matrix implies that the system of n+1 equations has either no solution or multiple solutions. The system of equations has no solution when these equations are contradictory to one another (insolvable conditions). On the other hand, if these equations are consistent (solvable conditions), the system of equations has multiple solutions. If the system of equations is insolvable, it means that the relative maximum does not exist. If the system of equations is solvable, multiple relative maximum of equal value exist.

2. Solvable conditions for n states of nature

It is found that the system of n+1 equations is solvable if and only if there are at most two distinct payoff values contained among n payoff values. This statement is proved as follows,

THEOREM I: A necessary and sufficient condition for Eqs. 3-15 and 3-2 to be solvable is that there are at most two distinct payoff values.

PROOF:

Case A: Only one payoff value exists, i.e., $X_i = X_1$ for all i.

Necessity: If Eqs. 3-15 and 3-2 are solvable, it is possible that $X_i = X_1$ for all i holds.

Sufficiency: If $X_i = X_1$ for all i (i=1, 2, ..., n), then

$$Y_i = \sum_{j=1}^{i} X_j = \sum_{j=1}^{i} X_1 = iX_1$$

$$Y_j = \sum_{s=1}^{j} X_j = \sum_{s=1}^{j} X_1 = jX_1$$

$$Z_i = \sum_{j=1}^{i} X_j^2 = \sum_{j=1}^{i} X_1^2 = iX_1^2$$

$$C_1 = \sum_{i=1}^{n} M_i\ X_i = \sum_{i=1}^{n} M_i\ X_1 = X_1 \sum_{i=1}^{n} M_i = X_1\ C_0$$

Substituting Y_i, Y_j, Z_i, and C_1 into Eq. 3-15 gives:

$$iX_1{}^2 - 2iX_1 \sum_{j=1}^{n} T_j(jX_1) - 2(X_1C_0)(iX_1) - i\lambda = 0$$

$$iX_1{}^2 - 2iX_1{}^2 \sum_{j=1}^{n} jT_j - 2iX_1{}^2\ C_0 - i\lambda = 0$$

$$X_1{}^2 - 2X_1{}^2 \sum_{j=1}^{n} jT_j - 2X_1{}^2\ C_0 - \lambda = 0$$

Subtracting $2X_1{}^2$ times Eq. 3-2,

$$X_1{}^2 - 2X_1{}^2 \sum_{j=1}^{n} jT_j - 2X_1{}^2\ C_0 - \lambda - 2X_1{}^2\ [1-C_0- \sum_{i=1}^{n} iT_i] = 0$$

$$X_1{}^2 - 2X_1{}^2 - \lambda = 0$$

$$\lambda = -X_1{}^2$$

Substituting the resulting λ into the simplified form of Eq. 3-15 yields:

$$X_1{}^2 - 2X_1{}^2 \sum_{j=1}^{n} jT_j - 2X_1{}^2\ C_0 - (-X_1{}^2) = 0$$

$$2X_1{}^2 - 2X_1{}^2 \sum_{j=1}^{n} jT_j - 2X_1{}^2\ C_0 = 0$$

$$2X_1{}^2\ [1 - \sum_{j=1}^{n} jT_j - C_0] = 0$$

$$1 - \sum_{j=1}^{n} jT_j - C_0 = 0$$

This equation is the final form for Eq. 3-15 which is identical to Eq. 3-2. Then n+1 equations have been reduced to one equation. Therefore, Eqs. 3-15 and 3-2 are solved with $\lambda = -X_1^2$ and $T_1, T_2, \ldots, T_n$ may be any positive value satisfying $\sum_{i=1}^{n} iT_i = 1 - C_0$.

<u>Case B</u>: Only two payoff values exist, i.e., $(X_i - X_1)(X_i - X_t) = 0$ for all i, where t is the smallest i value such that $X_i \neq X_1$. For $i = 1, 2, 3, \ldots, t-1$, substituting $Y_i = iX_1$ and $Z_i = iX_1^2$ into each of the first t-1 equations in Eq. 3-15 gives:

$$iX_1^2 - 2(iX_1)\sum_{j=1}^{n} T_jY_j - 2C_1(iX_1) - i\lambda = 0 \quad (i = 1, 2, .., t-1)$$

$$X_1^2 - 2X_1\sum_{j=1}^{n} T_jY_j - 2C_1X_1 - \lambda = 0 \qquad \text{(Eq. 3-16)}$$

Therefore, the first t-1 equations have been reduced to one equation, i.e., Eq. 3-16. Further, subtracting Eq. 3-16 times i from Eq. 3-15 for $i = t, t+1, \ldots, n$ gives:

$$Z_i - 2Y_i\sum_{j=1}^{n} T_jY_j - 2C_1Y_i - i\lambda - iX_1^2 + 2(iX_1)\sum_{j=1}^{n} T_jY_j + 2C_1(iX_1) + i\lambda = 0$$

$$(Z_i - iX_1^2) - 2(Y_i - iX_1)\sum_{j=1}^{n} T_jY_j - 2C_1(Y_i - iX_1) = 0$$

$$2(Y_i - iX_1)\left[\sum_{j=1}^{n} T_jY_j + C_1\right] = (Z_i - iX_1^2) \quad (i = t, t+1, \ldots, n)$$

$$\text{(Eq. 3-17)}$$

Now, the n+1 equations in Eqs. 3-15 and 3-2 have been reduced to a system of n-t+3 equations:

$$X_1^2 - 2X_1\sum_{j=1}^{n} T_jY_j - 2C_1X_1 - \lambda = 0 \qquad \text{(Eq. 3-16)}$$

$$2(Y_i - iX_1)\left[\sum_{j=1}^{n} T_jY_j + C_1\right] = (Z_i - iX_1^2) \qquad (i = t,\ t+1,\ \ldots,\ n)$$

(Eq. 3-17)

$$\sum_{i=1}^{n} iT_i = 1 - C_0$$

(Eq. 3-2)

<u>Necessity:</u> If Eqs. 3-16, 3-17, and 3-2 are solvable, then $Y_i - iX_1 \neq 0$ for $i=t,\ t+1,\ \ldots,\ n$. Since for any $i=t,\ t+1,\ \ldots,\ n$, if $Y_i - iX_1 = 0$, not only $Y_i = iX_1$, but also $Z_i = iX_1^2$ from Eq. 3-17. That is,

$$X_1 = \left[\sum_{j=1}^{i} X_j\right]/i \qquad \left(\text{since } Y_i = \sum_{j=1}^{i} X_j\right)$$

$$X_1^2 = \left[\sum_{j=1}^{i} X_j^2\right]/i \qquad \left(\text{since } Z_i = \sum_{j=1}^{i} X_j^2\right)$$

Hence,

$$\left\{\left[\sum_{j=1}^{i} X_j\right]/i\right\}^2 = \left[\sum_{j=1}^{i} X_j^2\right]/i$$

This implies that $X_i = X_1$ for all $j=1,\ 2,\ \ldots,\ i$, which contradicts the definition that $X_t \neq X_1$. Therefore, $Y_i - iX_1 \neq 0$ for any $i=t,\ t+1,\ \ldots,\ n$. As a result, Eq. 3-17 can be written as:

$$\sum_{j=1}^{n} T_jY_j + C_1 = (Z_i - iX_1^2)\ /\ 2(Y_i - iX_1) \quad (i = t,\ t+1,\ \ldots,\ n)$$

(Eq. 3-18)

Considering the case $i = t$, t is such that $X_j = X_1$ for $j=1,\ 2,\ \ldots,\ t-1$, and $X_t \neq X_1$. Then,

$$Y_t = \sum_{j=1}^{t} X_j = \sum_{j=1}^{t-1} X_j + X_t = \sum_{j=1}^{t-1} X_1 + X_t = (t-1)X_1 + X_t$$

$$Z_t = \sum_{j=1}^{t} X_j{}^2 = \sum_{j=1}^{t-1} X_j{}^2 + X_t{}^2 = \sum_{j=1}^{t-1} X_1{}^2 + X_t{}^2 = (t-1)X_1{}^2 + X_t{}^2$$

Substituting Y_t and Z_t into Eq. 3-18 and setting i=t,

$$\begin{aligned}\sum_{j=1}^{n} T_jY_j + C_1 &= (Z_t-tX_1{}^2) \;/\; 2(Y_t-tX_1)\\ &= [(t-1)X_1{}^2+X_t{}^2-tX_1{}^2] \;/\; 2[(t-1)X_1+X_t-tX_1]\\ &= [X_t{}^2 - X_1{}^2] \;/\; 2[X_t - X_1]\\ &= [X_t + X_1]/2\end{aligned}$$

$[\sum_{j=1}^{n} T_jY_j + C_1]$ is a constant that can be applied to any equation in Eq. 3-18.

$$[X_t+X_1]/2 = (Z_i-iX_1{}^2) \;/\; 2(Y_i-iX_1) \quad (i = t, t+1, \ldots, n)$$

$$(Z_i-iX_1{}^2) = (Y_i-iX_1)(X_t+X_1) \quad (i = t, t+1, \ldots, n)$$

Subtracting the equation for i-1 from each equation for i gives:

$$(Z_i-iX_1{}^2)-[Z_{i-1}-(i-1)X_1{}^2]=(X_t+X_1)\{(Y_i-iX_1)-[Y_{i-1}-(i-1)X_1]\} \quad (i = t+1, \ldots, n)$$

$$Z_i-Z_{i-1}-X_1{}^2 = (X_t+X_1)\{Y_i-Y_{i-1}-X_1\} \quad (i = t+1, \ldots, n)$$

$$\sum_{j=1}^{i} X_j{}^2 - \sum_{j=1}^{i-1} X_j{}^2 - X_1{}^2 = (X_t+X_1)\{\sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} X_j - X_1\}$$

$$X_i{}^2 - X_1{}^2 = (X_t+X_1)\{X_i - X_1\} \quad (i = t+1, \ldots, n)$$

$$(X_i + X_1)(X_i - X_1) = (X_t + X_1)(X_i - X_1) \quad (i = t+1, \ldots, n)$$

There are two cases that satisfy this equation, i.e.,

i. $X_i = X_1$

ii. $X_i \neq X_1$, then $X_i + X_1 = X_t + X_1$, $X_i = X_t$

Therefore, if Eqs. 3-16, 3-17, and 3-2 (equivalent to Eqs. 3-15 and 3-2) are solvable, $X_i = X_1$ for i=1, 2, ..., t-1, and $X_i = X_1$, or $X_i = X_t$ for i = t+1, .., n. This completes the proof of necessity.

Sufficiency: Suppose $(X_i - X_1)(X_i - X_t) = 0$ for i = 1, 2, .., n. That is, either $X_i = X_1$ or $X_i = X_t$ for i = 1, 2, .., n. Let a_i be defined as the number of occurrences that $X_j = X_1$ for j = 1, 2, .., i. Then Y_i and Z_i could be rewritten as:

$$Y_i = \sum_{j=1}^{i} X_j = \sum_{X_j=X_1} X_j + \sum_{X_j=X_t} X_j = a_i\ X_1 + (i-a_i)\ X_t$$

$$Z_i = \sum_{j=1}^{i} X_j^2 = \sum_{X_j=X_1} X_j^2 + \sum_{X_j=X_t} X_j^2 = a_i\ X_1^2 + (i-a_i)\ X_t^2$$

Substituting the expressions for Y_i and Z_i into Eq. 3-17,

$$2(Y_i-iX_1)[\sum_{j=1}^{n} T_jY_j + C_1] = (Z_i-iX_1^2) \quad (i = t, t+1, ..., n)$$

$$2[a_iX_1+(i-a_i)X_t-iX_1][\sum_{j=1}^{n} T_jY_j+C_1] = [a_iX_1^2+(i-a_i)X_t^2-iX_1^2] \quad (i = t, t+1, ..., n)$$

$$2[(i-a_i)X_t-(i-a_i)X_1][\sum_{j=1}^{n} T_jY_j+C_1] = [(i-a_i)X_t^2-(i-a_i)X_1^2] \quad (i = t, t+1, ..., n)$$

$$2(i-a_i)(X_t-X_1)[\sum_{j=1}^{n} T_jY_j+C_1] = (i-a_i)(X_t^2-X_1^2) \quad (i = t, t+1, ..., n)$$

$$2(i-a_i)(X_t-X_1)[\sum_{j=1}^{n} T_jY_j+C_1] = (i-a_i)(X_t+X_1)(X_t-X_1) \quad (i = t, t+1, ..., n)$$

Since $i-a_i \neq 0$ and $X_t \neq X_1$,

$$\sum_{j=1}^{n} T_j Y_j + C_1 = (X_t + X_1)/2 \qquad \text{(Eq. 3-19)}$$

Now the n+1 equations in Eqs. 3-15 and 3-2 have been reduced to three equations:

$$X_1^2 - 2X_1 \sum_{i=1}^{n} T_i Y_i - 2C_1 X_1 - \lambda = 0 \qquad \text{(Eq. 3-16)}$$

$$\sum_{i=1}^{n} T_i Y_i + C_1 = (X_t + X_1)/2 \qquad \text{(Eq. 3-19)}$$

$$\sum_{i=1}^{n} iT_i = 1 - C_0 \qquad \text{(Eq. 3-2)}$$

Substituting Eq. 3-19 into Eq. 3-16 yields:

$$X_1^2 - 2X_1 \left[\sum_{j=1}^{n} T_j Y_j + C_1\right] - \lambda = 0$$

$$X_1^2 - 2X_1 \left[(X_t + X_1)/2\right] - \lambda = 0$$

$$X_1^2 - X_1 X_t - X_1^2 - \lambda = 0$$

$$\lambda = -X_1 X_t$$

Therefore, the three equations (Eqs. 3-16, 3-19, and 3-2) are always solvable. This completes the proof of Theorem I.

3. Locations of relative maximums for 1 and 2 payoff values

Theorem I proved that if there are at most two distinct payoff values for n states of nature, it is necessary and sufficient for a relative maximum variance to exist. In other words, a relative maximum does not exist if there are more than two distinct payoff values. Therefore, for the case that there are more than two distinct payoff values, the global maximum variance must occur at the corner boundary points.

In the case of only one payoff value, any set of T values, $(T_1, T_2, \ldots, T_n)$, satisfying $\sum_{i=1}^{n} iT_i = 1 - C_0$ and $T_i \geq 0$ is one of the multiple solutions that result in the relative maximum variances. However, since there is only one payoff value, the variance is always equal to zero.

In the case of two distinct payoff values, the relative maximum variance can be located by solving Eqs. 3-19 and 3-2. By subtracting X_1 times Eq. 3-2 from Eq. 3-19, Eq. 3-19 becomes:

$$\sum_{i=1}^{n} T_i Y_i + C_1 - X_1 \sum_{i=1}^{n} iT_i = (X_t+X_1)/2 - X_1(1-C_0)$$

Substituting $Y_i = a_i X_1 + (i-a_i)X_t$,

$$\sum_{i=1}^{n} [a_i X_1+(i-a_i)X_t]T_i - \sum_{i=1}^{n} iX_1 T_i = (X_t+X_1)/2 - X_1 + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} \{[a_i X_1+(i-a_i)X_t]-iX_1\}\ T_i = (X_t+X_1-2X_1)/2 + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} \{(i-a_i)X_t-(i-a_i)X_1\}\ T_i = (X_t-X_1)/2 + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} \{(i-a_i)(X_t-X_1)\}\ T_i = (X_t-X_1)/2 + X_1 C_0 - C_1$$

Let $C' = \sum_{\substack{i \\ X_i=X_1}} M_i$, and $C'' = \sum_{\substack{i \\ X_i=X_t}} M_i$,

Then, $C_0 = \sum_{i=1}^{n} M_i = C' + C''$

$$C_1 = \sum_{i=1}^{n} M_i\ X_i = C'X_1 + C''X_t$$

$$C_2 = \sum_{i=1}^{n} M_i\ X_i^{\ 2} = C'X_1^{\ 2} + C''X_t^{\ 2}$$

Substituting the expression for C_0 and C_1 yields:

$$\sum_{i=1}^{n} (i-a_i)(X_t-X_1)T_i = (X_t-X_1)/2 + X_1(C'+C'') - C'X_1 - C''X_t$$

$$\sum_{i=1}^{n} (i-a_i)(X_t-X_1)T_i = (X_t-X_1)/2 - (X_t-X_1)C''$$

$$\sum_{i=1}^{n} (i-a_i)T_i = 1/2 - C'' \qquad \text{(since } X_t \neq X_1\text{)}$$

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' \qquad \text{(since } a_i = i \text{ for } i = 1, 2, \ldots, t-1\text{)}$$

Eq. 3-19 is replaced by:

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' \qquad \text{(Eq. 3-20)}$$

By subtracting Eq. 3-20 from Eq. 3-2, Eq. 3-2 becomes:

$$\sum_{i=1}^{n} iT_i - \sum_{i=t}^{n} (i-a_i)T_i = 1 - C_0 - 1/2 + C''$$

$$\sum_{i=1}^{t-1} iT_i + \sum_{i=t}^{n} iT_i - \sum_{i=t}^{n} iT_i + \sum_{i=t}^{n} a_iT_i = 1 - C' - C'' - 1/2 + C''$$

$$\sum_{i=1}^{t-1} iT_i + \sum_{i=t}^{n} a_iT_i = 1/2 - C'$$

Since $a_i = i$ for $i = 1, 2, \ldots, t-1$, Eq. 3-2 is replaced by:

$$\sum_{i=1}^{n} a_iT_i = 1/2 - C' \qquad \text{(Eq. 3-21)}$$

Eq. 3-3 remains as follows:

$T_i \geq 0$ (for $i = 1, 2, \ldots, n$) (Eq. 3-3)

If there are two distinct payoff values for n states of nature, any solution to Eqs. 3-20 and 3-21 results in a relative maximum. Since there are multiple solutions for Eqs. 3-20 and 3-21, multiple relative maximums can be obtained from Eqs. 3-20 and 3-21. However, with the constraints of Eq. 3-3, the multiple relative maximums may not be located inside the feasible region.

It is important to be able to identify the conditions for which at least one relative maximum exists inside the feasible region. Since $T_i \geq 0$, $a_i > 0$, and $(i-a_i) \geq 0$ for $i = 1, 2, \ldots, n$, the left hand side of both Eqs. 3-20 and 3-21 are greater than or equal to zero. The corresponding right hand side of these two equations must also be greater than or equal to zero. Therefore, $1/2 - C'' \geq 0$ and $1/2 - C' \geq 0$ are necessary to have at least one relative maximum inside the feasible region.

Despite the fact that the necessary conditions, $1/2 - C'' \geq 0$ and $1/2 - C' \geq 0$, are met, it is still possible that none of the multiple relative maximums exists inside the feasible region. In other words, the necessary conditions indicate that there is no relative maximum inside the feasible region if they are not met. But the fulfillment of the necessary conditions does not guarantee there is at least one relative maximum inside the feasible region.

A basic solution to Eqs. 3-20 and 3-21 has at most two unknowns which are not equal to zero. There are (n)(n-1)/2 basic solutions to Eqs. 3-20 and 3-21. Any linear combinations of these basic solutions is also a solution to Eqs. 3-20 and 3-21. Therefore, unless all the basic solutions are outside the feasible region, at least one relative maximum occurs inside the feasible region.

4. Common value of multiple relative maximums for two payoff values

In Section F.2, it was proved that no relative maximum exists if there are more than two distinct payoff values for n ($n \geq 3$) states of nature. If there is only one payoff value, the variance is always zero.

If there are only two distinct payoff values for n ($n \geq 3$) states of nature, the multiple relative maximums can be located by solving Eqs. 3-20, 3-21, and 3-3.

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' \qquad \text{(Eq. 3-20)}$$

$$\sum_{i=1}^{n} a_i T_i = 1/2 - C' \qquad \text{(Eq. 3-21)}$$

$$T_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \qquad \text{(Eq. 3-3)}$$

Although multiple relative maximums exist, they result in a common variance value. Therefore, any solution from Eqs. 3-20, 3-21, and 3-3 is sufficient to find the common value of the multiple relative maximums.

Since the system of equations has two equations and n ($n \geq 3$) unknowns, a basic solution to Eqs. 3-20 and 3-21 has at most two unknowns which are not equal to zero. It is then reasonable to assume that only T_1 and T_t are not equal to zero while setting all other unknowns equal to zero. With the above assumption, Eqs. 3-20, 3-21 and 3-3 can be rewritten as follows,

$$(t-a_t)T_t = 1/2 - C''$$

$$a_1T_1 + a_tT_t = 1/2 - C'$$

$$T_1 \geq 0, \quad T_t \geq 0$$

Since the payoff values for the states of nature from one to t-1 are all equal to X_1 according to the definition of t, then a_t = t-1. Therefore,

$$T_t = 1/2 - C'' \tag{Eq. 3-22}$$

$$\begin{aligned} T_1 &= 1/2 - C' - (t-1)T_t \\ &= 1/2 - C' - (t-1)/2 + (t-1)C'' \\ &= 1 - C' + (t-1)C'' - t/2 \end{aligned} \tag{Eq. 3-23}$$

The expressions for T_1 and T_t in Eqs. 3-22 and 3-23 represent one of the basic solutions to Eqs. 3-20 and 3-21. This solution is feasible if both T_1 and T_t are positive. The solution is infeasible, (in other words, it is outside the feasible region) if either T_1 or T_t is negative. Whether or not the resulting values of T_1 and T_t are feasible, the common variance value of the multiple relative maximums can be obtained from these values of T_1 and T_t.

From Eq. 3-1,

$$\begin{aligned} VAR &= C_2 + \sum_{i=1}^{n} T_iZ_i - C_1{}^2 - [\sum_{i=1}^{n} T_iY_i]^2 - 2C_1[\sum_{i=1}^{n} T_iY_i] \\ &= C_2 + \sum_{i=1}^{n} T_iZ_i - [C_1 + (\sum_{i=1}^{n} T_iY_i)]^2 \\ &= C_2 + T_1Z_1 + T_tZ_t - [C_1 + T_1Y_1 + T_tY_t]^2 \end{aligned}$$

Since $Y_1 = X_1$, $Y_t = (t-1)X_1 + X_t$, $Z_1 = X_1{}^2$, $Z_t = (t-1)X_1{}^2 + X_t{}^2$, $C_1 = C'X_1 + C''X_t$, $C_2 = C'X_1{}^2 + C''X_t{}^2$,

$$VAR = C_2 + T_1X_1{}^2 + T_t[(t-1)X_1{}^2+X_t{}^2] - \{C_1 + T_1X_1 + T_t[(t-1)X_1+X_t]\}^2$$

$$= C'X_1^2 + C''X_t^2 + T_1X_1^2 + T_t[(t-1)X_1^2+X_t^2]$$
$$- \{C'X_1 + C''X_t + T_1X_1 + T_t[(t-1)X_1+X_t]\}^2$$

Substitute the expression of T_1 and T_t into the above equation,

$$T_t = 1/2 - C'' \qquad \text{(Eq. 3-22)}$$

$$T_1 = 1 - C' + (t-1)C'' - t/2 \qquad \text{(Eq. 3-23)}$$

Then,

$$\begin{aligned}
\text{VAR} &= C'X_1^2 + C''X_t^2 + [1-C'+(t-1)C''-t/2]X_1^2 + (1/2-C'')[(t-1)X_1^2+X_t^2] \\
&\quad - \{C'X_1 + C''X_t + [1-C'+(t-1)C''-t/2]X_1 + (1/2-C'')[(t-1)X_1+X_t]\ \}^2 \\
&= C'X_1^2 + C''X_t^2 + X_1^2 - C'X_1^2 + (t-1)C''X_1^2 - tX_1^2/2 \\
&\qquad + (t-1)X_1^2/2 + X_t^2/2 - C''(t-1)X_1^2 - C''X_t^2 \\
&\quad - \{C'X_1 + C''X_t + X_1 - C'X_1 + (t-1)C''X_1 - tX_1/2 \\
&\qquad + (t-1)X_1/2 + X_t/2 - C''(t-1)X_1 - C''X_t\ \}^2 \\
&= X_1^2/2 + X_t^2/2 - \{\ X_1/2 + X_t/2\ \}^2 \\
&= X_1^2/2 + X_t^2/2 - X_1^2/4 - X_t^2/4 - X_1X_t/2 \\
&= (X_1^2 + X_t^2 - 2X_1X_t)\ /\ 4 \\
&= (X_t - X_1)^2\ /\ 4 \qquad \text{(Eq. 3-24)}
\end{aligned}$$

If there are only two distinct payoff values, Eq. 3-24 is the common variance value of the multiple relative maximum. Notice that this relative maximum variance value becomes the global maximum variance only if at least one relative maximum exists inside the feasible region.

5. Counter example

Up to this point, the search for the extreme variances under strict ranking has closely followed the pattern of the search for the extreme variances under weak ranking. However, a counter example can be developed to show the parallel cannot be taken further. Although the condition, that there are at most two distinct payoff values, is necessary and sufficient for the existence of relative maximums, it is no longer true that there will always be a corner boundary point of equal value.

Counter-example: Consider five possible states of nature and a strategy for which relative maximums exist inside the feasible region. The payoff values and the minimum differences between the states of nature are:

$X_1=10,\quad X_2=20,\quad X_3=20,\quad X_4=10,\quad X_5=10$

$k_1=.10,\quad k_2=.04,\quad k_3=.03,\quad k_4=.02,\quad k_5=.01$

Solution to the counter-example: By the definition of M_i, the minimum requirements of probabilities are:

$M_1=.20,\quad M_2=.10,\quad M_3=.06,\quad M_4=.03,\quad M_5=.01$

Since there are only two distinct payoffs, the coefficients C' and C'' are:

$$C' = \sum_{\substack{i \\ X_i=X_1}} M_i = M_1 + M_4 + M_5 = .24 \text{ and } C'' = \sum_{\substack{i \\ X_i=X_t}} M_i = M_2 + M_3 = .16$$

a. Relative maximums: Since the necessary conditions are met, the relative maximums can be located by using Eqs. 3-20 and 3-21.

From Eq. 3-20,

$$T_2 + 2T_3 + 2T_4 + 2T_5 = .5 - .16$$

From Eq. 3-21,

$$T_1 + T_2 + T_3 + 2T_4 + 3T_5 = .5 - .24$$

Solving these two equations simultaneously, one of the basic solutions for Eqs. 3-20 and 3-21 is:

$$T_1 = .09,\quad T_3 = .17,\quad T_2 = T_4 = T_5 = 0$$

Since this is a feasible solution, the relative maximum is a global maximum. The variance value of the global maximum can be calculated by Eq. 3-24.

$$VAR = (X_t - X_1)^2 / 4 = (20-10)^2 / 4 = 25.00$$

b. Variances at corner boundary points: Corner solutions occur when $T_i = (1-C_0)/i$ (i= 1, 2, 3, 4, or 5) with all other T_j ($j \neq i$) set at zero. The values of the variance at the corner solutions can then be evaluated and expressed in the following table,

Corner Solution	Resulting Probabilities P_1	P_2	P_3	P_4	P_5	Variance Value
1	.80	.10	.06	.03	.01	13.44
2	.50	.40	.06	.03	.01	24.84
3	.40	.30	.26	.03	.01	24.64
4	.35	.25	.21	.18	.01	24.84
5	.32	.22	.18	.15	.13	24.00

The resulting variance value from the global maximum lies outside the range 13.44 to 24.84 defined by the corner boundary points. The global maximal variance for this example is 25.00, and the global minimal variance 13.44.

6. Summary of solutions for n states of nature

Thus it may be concluded that the assessment of the extreme values of the variance under strict ranking must proceed in the following way.

a. Global minimum At least one global minimum occurs at the corner boundary points despite the number of distinct payoff values.

b. Global maximum

i. If there are more than two distinct payoff values, the global maximum must occur at corner boundary points.

ii. If there are only two distinct payoff values, it is possible that relative maximum exist inside the feasible region. Check the necessary conditions: 1) $1/2 - C'' \geq 0$, and 2) $1/2 - C' \geq 0$. If the necessary conditions are not met, the global maximum will still occur at corner boundary points. If the necessary conditions are met, Eqs. 3-20, 3-21, and 3-3 are served to locate the relative maximums. Check all basic solutions to Eqs. 3-20 and 3-21.

Case 1: If at least one basic solution of Eqs. 3-20 and 3-21 is feasible, the global maximal variance can be calculated by Eq. 3-24.

Case 2: If all the basic solutions to Eqs. 3-20 and 3-22 are not feasible, (for example, the value of X_3 is changed to 10 in the above counter example), the global maximum must occur at one of the corner boundary points.

IV. EXTREME INDEXES OF UTILITY UNDER STRICT RANKING

It is desirable to evaluate both the measure of central tendency (expected value) and the measure of dispersion (variance or standard variation) of a strategy. The algorithms of searching for 1) the extreme expected values under both weak and strict ranking, and 2) the extreme values of the variance under weak ranking were developed by Cannon and Kmietowicz [1974], Agunwamba [1980], and Kmietowicz and Pearman [1981]. The proof for determining the extreme values of the variance for strict ranking was developed in the previous chapter.

However, Kmietowicz and Pearman pointed out that

> "...it is possible that the sets of probabilities of states of nature underlying the two independent calculations (i.e., expected values and variance) need not be the same. Although it is quite plausible that the decision maker might approach strategy evaluation on the basis of independent calculations, it does suggest that he be required to maintain two mutually inconsistent views of the probability distribution of states of nature."

To avoid this difficulty, Kmietowicz and Pearman investigated incorporating the expected value and variance into a single index, index of utility, under weak ranking. They assumed that the expected value and the variance can be traded off linearly by a coefficient of risk aversion, b. Then the index of utility for a strategy can be written as:

$$I = EXP + b * VAR$$

where I = index of utility considering both the expected value and variance of a strategy
EXP = expected value of payoff values under a set of probabilities
b = coefficient of risk aversion which is a trade-off rate between the expected value and the variance
VAR = variance of payoff values under a set of probabilities

Kmietowicz and Pearman then developed an algorithm to search for the set of probabilities which gives the extreme indexes of utility under the conditions of weak ranking. They concluded that the assessment of the extreme indexes of utility which is a linear combination of expected value and variance must proceed in the following way:

1. If there are more than two distinct payoff values, the extreme indexes must occur at the corner boundary points.
2. If there are only two distinct payoff values, it is possible that the extreme indexes occur inside the feasible region. The extreme indexes can be identified by using equation 5.7 and 5.8 [Kmietowicz and Pearman 1981]. However, depending on the relationship between the coefficient of risk aversion and the two distinct payoff values, the identified extreme indexes need not always be located inside the feasible region. In the case that no extreme index can be identified inside the feasible region, the extreme indexes will still occur at the corner boundary points.

Another objective of this research is to search for a set of probability values under strict ranking that results in the extreme values (maximum and minimum values) of an index of utility. The index of utility is a linear combination of the expected value (mean) and the variance with a trade-off coefficient between the mean and the variance. The trade-off coefficient is referred to as the coefficient of risk aversion. A negative coefficient of risk aversion denotes an aversion to risk because a negative b value indicates an attempt to avoid risk by imposing a heavier penalty on a greater variance. As the value of b becomes increasingly negative, the more aversive to risk (conservative) the decision maker becomes. A positive coefficient of risk aversion denotes a preference to risk.

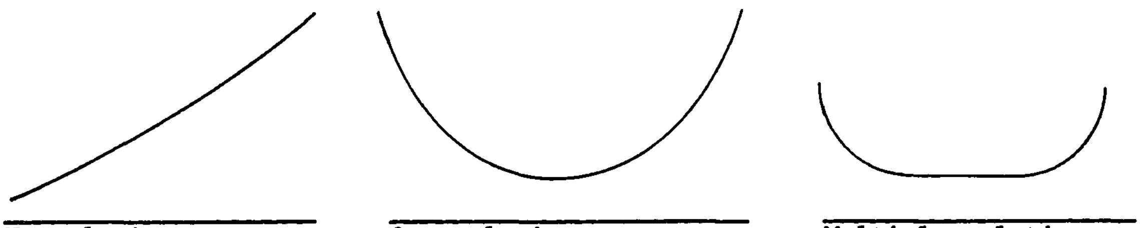
The resulting index of utility with a negative coefficient of risk aversion is a reasonable figure of merit for project evaluation. The index considers not only the expected returns from the project, but also is efficient in avoiding projects with great variation in return.

Section A defines the objective function and its constraints in searching for the extreme indexes of utility.

Section B examines the objective function and its constraints and recognizes it as a quadratic function subject to two linear constraints. The two linear constraints define the feasible region (combination of all feasible solutions) where the extreme indexes (maximum and minimum indexes) can lie.

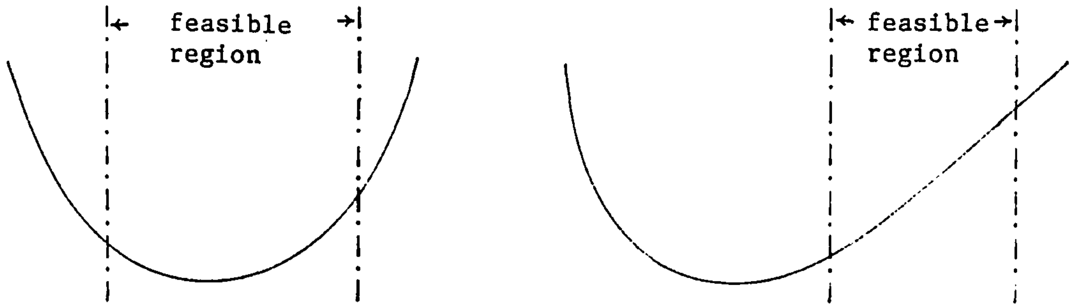
The quadratic term, because of the inherent negative feature of the coefficient of risk aversion, b, is positive. The positive quadratic term in the objective function dictates that the objective function is convex. Therefore, any relative extreme point must be a relative minimum. A relative maximum does not exist. As a result, the global maximum index of utility always occurs at the corner boundary points.

Section C describes the general approach that is used to search for the global minimum index of utility. Using a Lagrange multiplier and partial differentiation, a system of linear equations is formed. Any solution to the system of linear equations results in the relative minimum referred to in Section B. However, the system of equations may have 1) no solution indicating that no relative minimum exists, 2) one single solution (one relative minimum), or 3) multiple solutions (multiple relative minimums) of equal value.



No solution One solution Multiple solutions

If a relative minimum formed by the convex function exists within the feasible region, it is also a global minimum. If no relative minimum exists, or the relative minimum(s) of the convex function lies outside of the feasible region, then the global minimum occurs at one of the corner boundary points of the feasible region.



Relative minimum occurring inside of the feasible region (relative minimum = global minimum)

Relative minimum occurring outside of the feasible region (relative minimum ≠ global minimum)

In Section D, the objective function for the index of utility takes into consideration two possible states of nature. By applying the solving algorithm of the Lagrange multiplier and partial differentiation, it is possible to solve for the two T values. Here it is possible to obtain positive as well as negative T values. If both T values are positive, the resulting relative minimum is located inside of the feasible region. This relative minimum is a global minimum. If either one of the two T value is negative, the resulting relative minimum lies outside of the feasible region. Hence, when a negative T value occurs, the global minimum must occur at one of the corner boundary points of the feasible region.

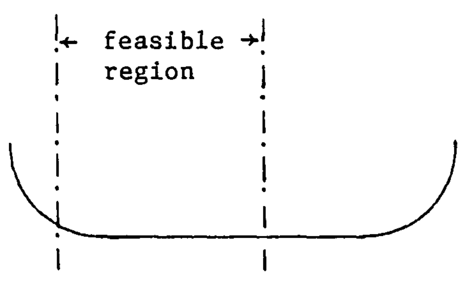
In Section D, a numerical example demonstrates the procedure for locating the relative minimum. The positive T values obtained indicate that the resulting relative minimum index lies inside the feasible region. Therefore, the relative minimum is the global minimum index. And the value of the global minimum index lies outside the index range as defined by the corner boundary points.

In Section E, the objective function for the index of utility takes into consideration more than two states of nature. Using the same solving algorithm of the Lagrange multiplier and partial differentiation, a system of n+1 equations is formed. The determinant of the coefficient matrix for the system of equations is found to be zero. Therefore, the system of equations has either no solution if the equations are contradictory to one another (insolvable conditions), or multiple solutions if the equations are consistent (solvable condition).

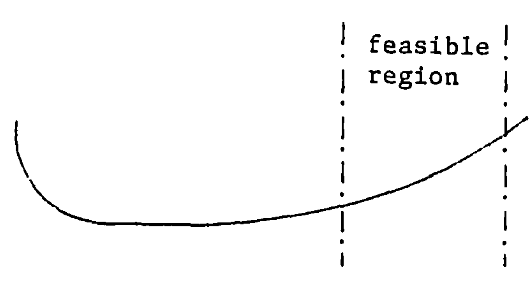
Solvable conditions for n states of nature are then dictated by Theorem II. For the system of equations to be solvable, Theorem II proves that there can be at most two distinct payoff values. Therefore, the global minimum must occur at the corner boundary points if there are more than two distinct payoff values. If there is only one payoff value, the index of utility is equal to the payoff value because the variance is always equal to zero.

If there are two distinct payoff values, multiple relative minimums of equal value can exist. The necessary conditions for the existence of at least one relative minimum inside the feasible region are developed. If the necessary conditions are not met, the global minimum index of utility must occur at one of the corner boundary points.

If the necessary conditions are met, it is possible that a relative minimum(s) exists inside the feasible region. The multiple relative minimums can be located by a pair of linear equations. Any solution to the pair of linear equations results in a relative minimum. The multiple relative minimums have a common value of the variance which can be calculated directly. However, only the relative minimum(s) inside the feasible region defines the global minimum(s). In case that none of the multiple relative minimums is located inside the feasible region, the global minimum index of utility must occur at one of the corner boundary points.



At least one of the multiple relative minimums of equal value exists inside of the feasible region.



All multiple relative minimums of equal value exist outside of the feasible region.

A numerical example is provided to demonstrate the procedure for searching for the multiple relative minimum points.

A. Objective Function and Constraints

Using the same notation from Chapter III, the formula for calculating the expected value under strict ranking is:

$$EXP = \sum_{i=1}^{n} P_i * X_i$$

$$= \sum_{i=1}^{n} (M_i + D_i) * X_i$$

$$= \sum_{i=1}^{n} M_i X_i + \sum_{i=1}^{n} D_i X_i$$

$$= C_1 + \sum_{i=1}^{n} T_i Y_i$$

And the expression for the variance is repeated as:

$$VAR = C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1 [\sum_{i=1}^{n} T_i Y_i]$$

Since the constraints remain the same as in Chapter III, the objective function and the constraints used to search for the extreme indexes of utility under strict ranking can be written as:

<u>Maximizing or Minimizing</u>

$$I = EXP + b * VAR \qquad \text{(Eq. 4-1)}$$

$$= C_1 + \sum_{i=1}^{n} T_i Y_i + b \{C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1 [\sum_{i=1}^{n} T_i Y_i]\}$$

Subject to

$$\sum_{i=1}^{n} i * T_i = 1 - C_0 \qquad \text{(Eq. 4-2)}$$

$$T_i \geq 0 \quad (\text{for } i = 1, 2, 3, \ldots, n) \qquad \text{(Eq. 4-3)}$$

Notice that all of the k_i´s are set equal to zero under weak ranking. Hence, the minimum requirement of probabilities, M_i, for all states of nature are zero. As a result, the constants (C_0, C_1, and C_2) which are composed of M_i are equal to zero. Therefore, Eq. 4-1 can be rewritten as the following equation for weak ranking.

$$I = \sum_{i=1}^{n} T_i Y_i + b \{ \sum_{i=1}^{n} T_i Z_i - [\sum_{i=1}^{n} T_i Y_i]^2 \}$$

The above equation is the same as the objective function under weak ranking derived by Kmietowicz and Pearman [1981]. Therefore, searching for the extreme indexes of utility under weak ranking is a special case of the problem under strict ranking where all of the k_i´s ($i = 1, 2, .., n$) are set equal to zero.

B. Nature of Objective Function and Constraints

Since the objective function is a quadratic function and the constraints are linear, the problem to be solved is in the form of quadratic programming. Since the constraints are the same as the case of searching for the extreme variances, the resulting feasible region also remains the same. The feasible region is a closed convex set formed by the intersection of a hyperplane and n closed positive half-spaces.

The objective function is formed by one quadratic term, $-b[\sum_{i=1}^{n} T_i Y_i]^2$, together with linear terms and constants. The coefficient of risk aversion, b, is always less than or equal to zero in any sound economic decision. (The non-positivity feature of the coefficient of risk aversion will be discussed in detail in Chapter V.) Therefore, the quadratic term is always greater than or equal to zero, and is defined as positive semidefinite. Any positive semidefinite quadratic form is a convex function over all of E^n [Theorem 3.4, Simmons 1975]. In addition, the linear terms are both convex and concave over all of E^n [Theorem 3.1, Simmons 1975]. The sum of two or more convex functions is convex [Theorem 3.3, Simmons 1975]. Therefore, the objective function, which is the sum of one convex quadratic term and several linear terms, is a convex function over all of E^n. As a result, a relative extreme point must be a relative minimum if such a relative extreme point exists.

C. Approach to Search for Extreme Indexes of Utility

Since the objective function of the index of utility is convex over all of E^n, the global minimum of the index of utility over the feasible region may either be determined by the relative minimum(s) if it exists inside the feasible region, or be located at corner boundary points of the feasible region. The global maximum of the index of utility over the feasible region must occur at the corner boundary points because no relative maximum exists.

To locate the relative minimum for the constrained quadratic programming problem, the method of Lagrange multipliers can be used. By introducing a Lagrange multiplier, the objective function is combined with the constraint (not including the non-negativity constraints) into a Lagrange function.

Setting the first partial derivatives of the Lagrange function equal to zero, a system of n+1 linear equations of n variables (T_i's) and the Lagrange multiplier is generated. A solution that satisfies the system of n+1 equations results in a relative minimum of the original quadratic programming problem. Since there are n+1 equations for n+1 unknowns (T_i's and the Lagrange multiplier), only one relative minimum exists in E^n if the n+1 linear equations are independent of one another. However, if some of the n+1 equations are dependent on one another (in which case the determinant of the coefficient matrix will be equal to zero), either no relative minimum or multiple relative minimums exist in E^n. Since the Lagrange function ignores the non-negativity constraints, the relative minimum(s) may be located outside the feasible region. Therefore, the search for the global minimum of the index of utility comprises the following three situations:

a) If at least one relative minimum exists within the feasible region, the constrained relative minimum(s) must be a global minimum(s) over the feasible region [Theorem 3.7, Simmons 1975].

b) If a relative minimum(s) exists, but the relative minimum(s) is located outside the feasible region, then the global minimums must occur at the corner boundary points.

c) If there is no relative minimum, then the global minimum must occur at the corner boundary points.

D. Extreme Indexes of Utility for Two States of Nature

The objective function and its constraints for a problem with two possible states of nature (two dimensional problem) are:

Maximizing or minimizing

$$I = C_1+T_1Y_1+T_2Y_2 + b\ \{C_2+T_1Z_1+T_2Z_2-C_1{}^2-[T_1Y_1+T_2Y_2]^2-2C_1(T_1Y_1+T_2Y_2)\}$$

Subject to

$$T_1 + 2T_2 = 1 - C_0$$

$$T_1 \geq 0,\ T_2 \geq 0$$

Since the objective function is convex, the global maximum must occur at the corner boundary points. In order to determine the global minimum, it is necessary to ascertain whether or not a relative minimum exists inside the feasible region.

1. Lagrange function

In the Lagrange multiplier method of searching for relative extreme points, it is not possible to include non-negativity constraints. Hence, non-negativity constraints are temporarily ignored. The usual procedure is to examine the solution in order to ascertain that the non-negativity constraints are satisfied.

Temporarily ignoring the non-negativity constraints, a standard Lagrange function can be formed to search for the extreme point (maximum or minimum index). The Lagrange function is:

$$L = C_1+T_1Y_1+T_2Y_2 + b\ \{C_2+T_1Z_1+T_2Z_2-C_1{}^2-[T_1Y_1+T_2Y_2]^2-2C_1(T_1Y_1+T_2Y_2)\}$$
$$+ \psi\ [1-C_0-T_1-2T_2]$$
$$= C_1+T_1Y_1+T_2Y_2 + b\ \{C_2+T_1Z_1+T_2Z_2-C_1{}^2-T_1{}^2Y_1{}^2-T_2{}^2Y_2{}^2-2T_1T_2Y_1Y_2$$
$$-2C_1T_1Y_1-2C_1T_2Y_2\} + \psi[1-C_0-T_1-2T_2]$$

For a relative minimum to exist, it is necessary that the three simultaneous equations formed from the partial derivatives of the Lagrange function with respect to T_1, T_2, and ψ be solvable [Schmidt 1974].

$$\frac{\partial L}{\partial T_1} = Y_1 + b\ \{Z_1 - 2T_1Y_1{}^2 - 2T_2Y_1Y_2 - 2C_1Y_1\} - \psi = 0 \qquad \text{(Eq. 4-4)}$$

$$\frac{\partial L}{\partial T_2} = Y_2 + b\ \{Z_2 - 2T_2Y_2{}^2 - 2T_1Y_1Y_2 - 2C_1Y_2\} - 2\psi = 0 \qquad \text{(Eq. 4-5)}$$

$$\frac{\partial L}{\partial \psi} = 1 - C_0 - T_1 - 2T_2 = 0 \qquad \text{(Eq. 4-6)}$$

2. <u>Solution for relative minimum index of utility</u>

Multiplied by (Y_2/Y_1), Eq. 4-4 becomes:

$$Y_2 + b\ \{Z_1(Y_2/Y_1) - 2T_1Y_1Y_2 - 2T_2Y_2{}^2 - 2C_1Y_2\} - \psi(Y_2/Y_1) = 0$$

Subtract the above equation from Eq. 4-5,

$$b\ \{Z_2 - Z_1(Y_2/Y_1)\} - \psi(2 - Y_2/Y_1) = 0$$

Multiplying by Y_1,

$$b\ \{Z_2Y_1 - Z_1Y_2\} - \psi(2Y_1 - Y_2) = 0$$

Therefore, the value for ψ is obtained:

$$\psi = b\ (Z_2Y_1 - Z_1Y_2)\ /\ (2Y_1 - Y_2) \qquad \text{(Eq. 4-7)}$$

Substitute the expression for ψ (Eq. 4-7), and $T_1 = 1 - C_0 - 2T_2$ (Eq. 4-6) into Eq. 4-5, and solve for T_2.

$$Y_2 + b\,\{Z_2-2T_2Y_2^2-2(1-C_0-2T_2)Y_1Y_2-2C_1Y_2\} - 2b(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2) = 0$$

$$Y_2 + b\{Z_2-2T_2Y_2^2-2(1-C_0)Y_1Y_2+4T_2Y_1Y_2-2C_1Y_2-2(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2)\} = 0$$

$$T_2b(2Y_2^2-4Y_1Y_2) = Y_2 + b\{Z_2-2(1-C_0)Y_1Y_2-2C_1Y_2-2(Z_2Y_1-Z_1Y_2)/(2Y_1-Y_2)\}$$

Substitute $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Z_1 = X_1^2$, and $Z_2 = X_1^2 + X_2^2$ into the equation,

$$\begin{aligned}T_2b[2(X_1+X_2)^2-4X_1(X_1+X_2)] = (X_1+X_2) + b\,\{&(X_1^2+X_2^2)-2(1-C_0)X_1(X_1+X_2)\\ &-2C_1(X_1+X_2)\\ &-2[(X_1^2+X_2^2)X_1-X_1^2(X_1+X_2)]/[2X_1-(X_1+X_2)]\}\end{aligned}$$

$$\begin{aligned}T_2b[2X_1^2+2X_2^2+4X_1X_2-4X_1^2-4X_1X_2] = X_1+X_2 + b\,\{&X_1^2+X_2^2-2X_1^2-2X_1X_2\\ &+2C_0X_1(X_1+X_2)-2C_1(X_1+X_2)\\ &-2[X_1^3+X_1X_2^2-X_1^3-X_1^2X_2]/[2X_1-X_1-X_2]\}\end{aligned}$$

$$\begin{aligned}T_2b[2X_2^2-2X_1^2] = X_1+X_2 + b\,\{&X_2^2-X_1^2-2X_1X_2+2C_0X_1(X_1+X_2)-2C_1(X_1+X_2)\\ &-2[X_1X_2^2-X_1^2X_2]/[X_1-X_2]\}\end{aligned}$$

$$2T_2b[X_2^2-X_1^2] = X_1+X_2 + b\{X_2^2-X_1^2-2X_1X_2+2C_0X_1(X_1+X_2)-2C_1(X_1+X_2)+2X_1X_2\}$$

$$2T_2b[X_2^2-X_1^2] = X_1+X_2 + b\{(X_2^2-X_1^2)+2C_0X_1(X_1+X_2)-2C_1(X_1+X_2)\}$$

$$\begin{aligned}T_2 = {} & (X_1+X_2)/2b(X_2^2-X_1^2)\\ & + b\{(X_2^2-X_1^2)+2C_0X_1(X_1+X_2)-2C_1(X_1+X_2)\}/2b(X_2^2-X_1^2)\end{aligned}$$

$$T_2 = 1/2b(X_2-X_1) + 1/2 + C_0X_1/(X_2-X_1) - C_1/(X_2-X_1)$$

Recall the definition for C_0 and C_1,

$$C_0 = 1k_1 + 2k_2$$

$$C_1 = M_1X_1 + M_2X_2 = (k_1+k_2)X_1 + k_2X_2$$

Substituting C_0 and C_1, the expression for T_2 can be further simplified.

$$\begin{aligned}T_2 &= 1/2b(X_2-X_1) + 1/2 + (k_1+2k_2)X_1/(X_2-X_1) - [(k_1+k_2)X_1+k_2X_2]/(X_2-X_1)\\ &= 1/2b(X_2-X_1) + 1/2 + [k_1X_1+2k_2X_1-k_1X_1-k_2X_1-k_2X_2]/(X_2-X_1)\\ &= 1/2b(X_2-X_1) + 1/2 + [k_2X_1-k_2X_2]/(X_2-X_1)\end{aligned}$$

$= 1/2b(X_2-X_1) + 1/2 - [k_2(X_2-X_1)]/(X_2-X_1)$

$$= 1/2b(X_2-X_1) + 1/2 - k_2 \qquad \text{(Eq. 4-8)}$$

Since $T_1 + 2T_2 = 1 - C_0$,

$T_1 = 1 - C_0 - 2T_2$

$= 1 - k_1 - 2k_2 - 1/b(X_2-X_1) - 1 + 2k_2$

$$= 1/b(X_1-X_2) - k_1 \qquad \text{(Eq. 4-9)}$$

Note that a feasible solution for T_1 and T_2 resulted in terms of b, X_1, X_2, k_1, and k_2. As long as the resulting values of T_1 and T_2 are positive, a feasible solution can be obtained. However, a negative value of T_1 or T_2 results in a solution that lies outside of the feasible region.

3. Counter example

Up to this point, the search for the relative minimum index of utility for two states of nature under strict ranking has closely followed the pattern of the relative maximum variances under strict ranking. However, a counter example can be developed to show that the relative minimum index of utility need not occur at the corner boundary points.

Counter-example: Consider a strategy under two possible states of nature. The payoff values and the minimum differences between the probabilities of the states of nature are:

$X_1=10, \quad X_2=20$

$k_1=.10, \quad k_2=.05$

And the coefficient of risk aversion is -0.25.

Solution to the counter-example: By the definition of M_i, the minimum requirements of probabilities are:

$M_1 = .15, \quad M_2 = .05, \quad C_0 = M_1 + M_2 = .20$

Relative minimum index of utility: The relative minimum index of utility can be located by using Eqs. 4-8 and 4-9.

From Eq. 4-9,

$$T_1 = 1/b(X_1 - X_2) - k_1 = 1/(-.25)(10-20) - .10 = .30$$

From Eq. 4-8,

$$T_2 = 1/2b(X_2 - X_1) + 1/2 - k_2 = 1/2(-.25)(20-10) + .5 - .05 = .25$$

Since this is a feasible solution as indicated by the positive T values, the resulting relative minimum is a global minimum index of utility. Then,

$D_1 = .55, \; D_2 = .25$

$P_1 = .70, \; P_2 = .30$

The resulting global minimum index of utility is 7.75 with a mean of 13.00 and variance of 21.00.

Indexes of utility at corner boundary points: The two corner boundary points are $T_i = (1-C_0)/i$ ($i = 1$ or 2) with the other T_j ($j \neq i$) set at zero. The values of the mean, variance, and index of utility at the corner solutions can be evaluated and expressed in the following table,

Corner boundary point	Probabilities P_1	Probabilities P_2	Mean Value	Variance Value	Index of Utility
1	.95	.05	10.50	4.75	9.3125
2	.55	.45	14.50	24.75	8.3125

The global minimum index of utility lies outside the range of 8.3125 to

9.3125 as defined by the corner boundary points. Therefore, the global minimum index for this example is 7.75; the global maximum index is 9.3125.

Thus, it may be concluded that the assessment of the extreme indexes of utility for two states of nature under strict ranking must proceed in the following way:

1. If there is only one payoff value, i.e., $X_1 = X_2$, the decision problem becomes trivial.
2. If there are two distinct payoff values, the global maximum index of utility always occurs at the corner boundary points. Eqs. 4-8 and 4-9 are served to locate the relative minimum index of utility. Depending upon the values assigned to b, X_1, X_2, and k_1, it need not always be the case that the relative minimum index is located inside the feasible region. If a relative minimum exists inside the feasible region, it is the global minimum index of utility. If the relative minimum index is located outside the feasible region (for example, the value of T_1 or T_2 is negative), the global minimum index of utility will be found at the corner boundary points of the feasible region.

E. Extreme Indexes of Utility for N States of Nature

The objective function and constraints in the search for the extreme indexes of utility for n states of nature (where $n \geq 3$) under strict ranking are:

Maximizing or minimizing

$$I = EXP + b * VAR \qquad \text{(Eq. 4-1)}$$

$$= C_1 + \sum_{i=1}^{n} T_i Y_i + b\ \{C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1[\sum_{i=1}^{n} T_i Y_i]\}$$

Subject to

$$\sum_{i=1}^{n} i * T_i = 1 - C_0 \qquad \text{(Eq. 4-2)}$$

$$T_i \geq 0 \quad \text{(for } i = 1, 2, 3, \ldots, n) \qquad \text{(Eq. 4-3)}$$

Since the objective function is convex, no relative maximum exists. The global maximum must occur at the corner boundary points. In order to determine the global minimum, the objective is to ascertain if a relative minimum exists inside the feasible region. A Lagrange function is used to search for the relative minimum(s).

1. Lagrange function

The appropriate Lagrange function for n states of nature is:

$$L = C_1 + \sum_{i=1}^{n} T_i Y_i + b\ \{C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1[\sum_{i=1}^{n} T_i Y_i]\}$$

$$+ \psi\ [1 - C_0 - \sum_{i=1}^{n} i\ T_i]$$

For a relative minimum to exist, it is necessary that the n+1 simultaneous linear equations formed from the partial derivatives of the Lagrange function with respect to the T_i values ($i = 1, 2, 3, \ldots, n$) and to ψ, be solvable for T_i and ψ [Schmidt 1974]. That is, it must be possible to solve the following n+1 equations:

$$\frac{\partial L}{\partial T_i} = Y_i + b\ \{Z_i - 2Y_i \sum_{j=1}^{n} T_j Y_j - 2C_1 Y_i\} - i\psi = 0 \quad (i = 1, 2, .., n)$$

(Eq. 4-10)

$$\frac{\partial L}{\partial \psi} = 1 - C_0 - \sum_{i=1}^{n} iT_i = 0 \quad \text{or} \quad \sum_{i=1}^{n} iT_i = 1 - C_0$$

(Eq. 4-2)

Eqs. 4-10 and 4-2 can be written in the form of matrixes:

$$\begin{bmatrix} \frac{\partial L}{\partial T_1} \\ \frac{\partial L}{\partial T_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial L}{\partial T_n} \\ \frac{\partial L}{\partial \psi} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \\ 0 \end{bmatrix} + \begin{bmatrix} bZ_1 \\ bZ_2 \\ \cdot \\ \cdot \\ \cdot \\ bZ_n \\ 1 \end{bmatrix} - \begin{bmatrix} 2bY_1^2 & 2bY_1Y_2 & .. & 2bY_1Y_n & 1 \\ 2bY_1Y_2 & 2bY_2^2 & .. & 2bY_2Y_n & 2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 2bY_1Y_n & 2bY_2Y_n & .. & 2bY_n^2 & n \\ 1 & 2 & .. & n & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_n \\ \psi \end{bmatrix} - \begin{bmatrix} 2bC_1Y_1 \\ 2bC_1Y_2 \\ \cdot \\ \cdot \\ \cdot \\ 2bC_1Y_n \\ C_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

Rearrange the matrix equation,

$$\begin{bmatrix} 2bY_1^2 & 2bY_1Y_2 & \cdots & 2bY_1Y_n & 1 \\ 2bY_1Y_2 & 2bY_2^2 & \cdots & 2bY_2Y_n & 2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 2bY_1Y_n & 2bY_2Y_n & \cdots & 2bY_n^2 & n \\ 1 & 2 & \cdots & n & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_n \\ \psi \end{bmatrix} = \begin{bmatrix} Y_1 + bZ_1 - 2bC_1Y_1 \\ Y_2 + bZ_2 - 2bC_1Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n + bZ_n - 2bC_1Y_n \\ 1 - C_0 \end{bmatrix}$$

Divide the first n rows by $2Y_1$, $2Y_2$, ..., and $2Y_n$, respectively,

$$\begin{bmatrix} bY_1 & bY_2 & \cdots & bY_n & 1/2Y_1 \\ bY_1 & bY_2 & \cdots & bY_n & 2/2Y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ bY_1 & bY_2 & \cdots & bY_n & 3/2Y_n \\ 1 & 2 & \cdots & n & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \\ \psi \end{bmatrix} = \begin{bmatrix} (1/2)+(bZ_1/2Y_1)-bC_1 \\ (1/2)+(bZ_2/2Y_2)-bC_1 \\ \vdots \\ (1/2)+(bZ_n/2Y_n)-bC_1 \\ 1-C_0 \end{bmatrix}$$

Subtract rows 2, 3, ..., n by row 1, respectively.

$$\begin{bmatrix} bY_1 & bY_2 & \cdots & bY_n & 1/2Y_1 \\ 0 & 0 & \cdots & 0 & (2/2Y_2)-(1/2Y_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & (n/2Y_n)-(1/2Y_1) \\ 1 & 2 & \cdots & n & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \\ \psi \end{bmatrix} = \begin{bmatrix} (1/2)+(bZ_1/2Y_1)-bC_1 \\ (bZ_2/2Y_2)-(bZ_1/2Y_1) \\ \vdots \\ (bZ_n/2Y_n)-(bZ_1/2Y_1) \\ 1-C_0 \end{bmatrix}$$

The determinant of the (n+1) by (n+1) coefficient matrix must be equal to zero [Cofactor method of calculating the determinant, Schmidt 1974]. The singular coefficient matrix implies that the system of n+1 equations has either no solution or infinite solutions. The system of equations has no solution when these equations are contradictory to one another (insolvable conditions). On the other hand, if these equations are consistent (solvable conditions), the system of equations has multiple solutions. If the system of equations is insolvable, it means that the relative minimum does not exist. If the system of equations is solvable, multiple relative minimums of equal value exist.

2. Solvable conditions for n states of nature

It is found that the system of n+1 equations is solvable if and only if there are at most two distinct payoff values contained among n payoff values. This statement is proved as follows,

THEOREM II: A necessary and sufficient condition for Eqs. 4-10 and 4-2 to be solvable is that there are at most two distinct payoff values.

PROOF:

Case A: Only one payoff value exists, i.e., $X_i = X_1$ for all i.

Necessity: If Eqs. 4-10 and 4-2 are solvable, it is possible that $X_i = X_1$ for all i holds.

Sufficiency: If $X_i = X_1$ for all i (i=1, 2, ..., n), then,

$$Y_i = iX_1$$

$$Y_j = jX_1$$

$$Z_i = iX_1^2$$

$$C_1 = X_1 C_0$$

Substituting Y_i, Y_j, Z_i, and C_1 into Eq. 4-10 gives:

$$iX_1 + b \{iX_1^2 - 2iX_1 \cdot \sum_{j=1}^{n} T_j(jX_1) - 2(X_1C_0)(iX_1)\} - i\psi = 0$$

$$iX_1 + b \{iX_1^2 - 2iX_1^2 \sum_{j=1}^{n} jT_j - 2iX_1^2C_0\} - i\psi = 0$$

$$X_1 + b \{X_1^2 - 2X_1^2 \sum_{j=1}^{n} jT_j - 2X_1^2C_0\} - \psi = 0$$

Subtracting $2bX_1^2$ times Eq. 4-2,

$$X_1 + b\{X_1^2-2X_1^2 \sum_{j=1}^{n} jT_j-2X_1^2C_0\} - \psi - 2bX_1^2[1-C_0-\sum_{i=1}^{n} iT_i] = 0$$

$$X_1 + b\ \{X_1^2 - 2X_1^2\} - \psi = 0$$

$$\psi = X_1 - bX_1^2$$

Substituting the resulting ψ into the simplified Eq. 4-10 yields:

$$X_1 + b\ \{X_1^2 - 2X_1^2 \sum_{j=1}^{n} jT_j - 2X_1^2C_0\} - [X_1 - bX_1^2] = 0$$

$$b\ \{2X_1^2 - 2X_1^2 \sum_{j=1}^{n} jT_j - 2X_1^2C_0\} = 0$$

$$2bX_1^2\ [1 - \sum_{j=1}^{n} jT_j - C_0] = 0$$

$$1 - \sum_{j=1}^{n} jT_j - C_0 = 0$$

This equation is the final form for Eq. 4-10 which is identical to Eq. 4-2. The n+1 equations have been reduced to one equation. Therefore, Eqs. 4-10 and 4-2 are solved with $\psi = X_1 - bX_1^2$ and T_1, T_2, ..., T_n may be any positive value satisfying $\sum_{i=1}^{n} iT_i = 1 - C_0$.

Case B: Only two payoff values exist, i.e., $(X_i-X_1)(X_i-X_t) = 0$ for all i, where t is the smallest i value such that $X_i \neq X_1$. For i = 1, 2, 3, ..., t-1, substituting $Y_i = iX_1$ and $Z_i = iX_1^2$ into each of the first t-1 equations in Eq. 4-10 gives:

$$iX_1 + b\ \{iX_1^2 - 2(iX_1) \sum_{j=1}^{n} T_jY_j - 2C_1(iX_1)\} - i\psi = 0 \quad (i=1,\ 2, .., \ t-1)$$

$$X_1 + b\ \{X_1^2 - 2X_1 \sum_{j=1}^{n} T_jY_j - 2C_1X_1\} - \psi = 0 \qquad \text{(Eq. 4-11)}$$

Therefore, the first t-1 equations have been reduced to one equation, i.e., Eq. 4-11. Further, subtracting Eq. 4-11 times i from Eq. 4-10 for i = t, t+1, ... , n gives:

$$Y_i + b\,\{Z_i - 2Y_i \sum_{j=1}^{n} T_j Y_j - 2C_1 Y_i\} - i\psi$$

$$- iX1 - b\,\{iX_1{}^2 - 2(iX_1) \sum_{j=1}^{n} T_j Y_j - 2C_1(iX_1)\} + i\psi = 0$$

$$(Y_i - iX_1) + b\,\{(Z_i - iX_1{}^2) - 2(Y_i - iX_1) \sum_{j=1}^{n} T_j Y_j - 2C_1(Y_i - iX_1)\} = 0$$

$$2b(Y_i - iX_1)[\sum_{j=1}^{n} T_j Y_j + C_1] - (Y_i - iX_1) = b(Z_i - iX_1{}^2) \quad (i=t,\ t+1, .., n)$$

(Eq. 4-12)

Now, the n+1 equations in Eqs. 4-10 and 4-2 have been reduced to a system of n-t+3 equations:

$$X_1 + b\,\{X_1{}^2 - 2X_1 \sum_{j=1}^{n} T_j Y_j - 2C_1 X_1\} - \psi = 0 \qquad \text{(Eq. 4-11)}$$

$$2b(Y_i - iX_1)[\sum_{j=1}^{n} T_j Y_j + C_1] - (Y_i - iX_1) = b(Z_i - iX_1{}^2) \quad (i=t,\ t+1, .., n)$$

(Eq. 4-12)

$$\sum_{i=1}^{n} iT_i = 1 - C_0 \qquad \text{(Eq. 4-2)}$$

Necessity: If Eqs. 4-11, 4-12, and 4-2 are solvable, then $Y_i - iX_1 \neq 0$ for i = t, t+1, ..., n. Since for any i = t, t+1, ..., n, if $Y_i - iX_1 = 0$, not only $Y_i = iX_1$, but also $Z_i = iX_1{}^2$ from Eq. 4-12. This implies that $X_i = X_1$ for all i = t, t+1, ..., n, which contradicts the definition that $X_t \neq X_1$. Therefore, $Y_i - iX_1 \neq 0$

for any i=t, t+1, ..., n. As a result, Eq. 4-12 can be written as:

$$\sum_{j=1}^{n} T_j Y_j + C_1 = [b(Z_i - iX_1^2) + (Y_i - iX_1] \;/\; 2b(Y_i - iX_1) \quad (i=t, .., n)$$

(Eq. 4-13)

Considering the case i = t, t is such that $X_j = X_1$ for j=1, 2, ..., t-1, and $X_t \neq X_1$. Then,

$$Y_t = (t-1)X_1 + X_t$$

$$Z_t = (t-1)X_1^2 + X_t^2$$

Substituting Y_t and Z_t into Eq. 4-13 setting i=t,

$$\begin{aligned}\sum_{j=1}^{n} T_j Y_j + C_1 &= [b(Z_t - tX_1^2) + (Y_t - tX_1] \;/\; 2b(Y_t - tX_1) \\ &= \frac{\{b[(t-1)X_1^2 + X_t^2 - tX_1^2] + [(t-1)X_1 + X_t - tX_1]\}}{2b[(t-1)X_1 + X_t - tX_1]} \\ &= \{b[X_t^2 - X_1^2] + [X_t - X_1]\} \;/\; 2b[X_t - X_1] \\ &= [b(X_t + X_1) + 1] \;/\; 2b\end{aligned}$$

$[\sum_{j=1}^{n} T_j Y_j + C_1]$ is a constant that can be applied to any equation in Eq. 4-13.

$$[b(X_t + X_1) + 1]/2b = [b(Z_i - iX_1^2) + (Y_i - iX_1)] \;/\; 2b(Y_i - iX_1)$$

(i = t, t+1, ..., n)

$$[b(X_t + X_1) + 1](Y_i - iX_1) = [b(Z_i - iX_1^2) + (Y_i - iX_1)] \quad (i = t, .., n)$$

Subtracting the equation for i-1 from each equation for i gives:

$$\begin{aligned}&[b(X_t + X_1) + 1](Y_i - iX_1) - [b(X_t + X_1) + 1][Y_{i-1} - (i-1)X_1] \\ &= [b(Z_i - iX_1^2) + (Y_i - iX_1)] - \{b[Z_{i-1} - (i-1)X_1^2] + [Y_{i-1} - (i-1)X_1]\}\end{aligned}$$

(i = t+1, ..., n)

$$[b(X_t+X_1)+1][Y_i-iX_1-Y_{i-1}+(i-1)X_1]$$

$$= b[Z_i-iX_1{}^2-Z_{i-1}+(i-1)X_1{}^2] + [Y_i-iX_1-Y_{i-1}+(i-1)X_1]$$

$$(i = t+1, \ldots, n)$$

$$[b(X_t+X_1)+1][Y_i-Y_{i-1}-X_1] = b[Z_i-Z_{i-1}-X_1{}^2]+[Y_i-Y_{i-1}-X_1]$$

$$(i = t+1, \ldots, n)$$

$$[b(X_t+X_1)+1][X_i-X_1] = b[X_i{}^2-X_1{}^2]+[X_i-X_1] \quad (i = t+1, \ldots, n)$$

$$[b(X_t+X_1)+1](X_i-X_1) = b(X_i+X_1)(X_i-X_1)+(X_i-X_1)$$

$$(i = t+1, \ldots, n)$$

$$[b(X_t+X_1)+1](X_i-X_1) = [b(X_i+X_1)+1](X_i-X_1) \quad (i = t+1, \ldots, n)$$

There are two cases that satisfy this equation, i.e.,

i. $X_i = X_1$

ii. $X_i \neq X_1$, then $b(X_t+X_1)+1 = b(X_i+X_1)+1$, $X_i = X_t$

Therefore, if Eqs. 4-11, 4-12, and 4-2 (equivalent to Eqs. 4-10 and 4-2) are solvable, $X_i = X_1$ for i=1, 2, ..., t-1, and $X_i = X_1$, or $X_i = X_t$ for i=t+1, ...,n. This completes the proof of necessity.

<u>Sufficiency</u>: Suppose $(X_i - X_1)(X_i - X_t) = 0$ for i = 1, 2, .., n. That is, either $X_i = X_1$ or $X_i = X_t$ for i = 1, 2, .., n. Let a_i be defined as the number of occurrences that $X_j = X_1$ for j = 1, 2, .., i. Then Y_i and Z_i could be rewritten as:

$$Y_i = a_i X_1 + (i-a_i) X_t$$

$$Z_i = a_i X_1{}^2 + (i-a_i) X_t{}^2$$

Then, $Y_i - iX_1 = a_i X_1 + (i-a_i) X_t - iX_1 = (i-a_i)(X_t-X_1)$

$$Z_i - iX_1{}^2 = a_i X_1{}^2 + (i-a_i) X_t{}^2 - iX_1{}^2 = (i-a_i)(X_t{}^2-X_1{}^2)$$

Substitute the expressions for Y_i-iX_1 and $Z_i-iX_1{}^2$ into Eq. 4-12.

$$2b(Y_i - iX_1)[\sum_{j=1}^{n} T_jY_j + C_1] - (Y_i - iX_1) = b(Z_i - iX_1^2) \quad (i=t,.., n)$$

$$2b(i-a_i)(X_t-X_1)[\sum_{j=1}^{n} T_jY_j+C_1]-(i-a_i)(X_t-X_1)=b(i-a_i)(X_t^2-X_1^2)$$
$$(i = t, t+1, ..., n)$$

$$2b(i-a_i)(X_t-X_1)[\sum_{j=1}^{n} T_jY_j+C_1]=b(i-a_i)(X_t^2-X_1^2)+(i-a_i)(X_t-X_1)$$
$$(i = t, t+1, ..., n)$$

$$2b(i-a_i)(X_t-X_1)[\sum_{j=1}^{n} T_jY_j+C_1] = (i-a_i)(X_t-X_1)[b(X_t+X_1)+1]$$
$$(i = t, t+1, ..., n)$$

Since $i-a_i \neq 0$ and $X_t \neq X_1$,

$$\sum_{j=1}^{n} T_jY_j + C_1 = [b(X_t+X_1)+1] /2b \qquad \text{(Eq. 4-14)}$$

Now the n+1 equations in Eqs. 4-10 and 4-2 have been reduced to three equations:

$$X_1 + b \{X_1^2 - 2X_1\sum_{j=1}^{n} T_jY_j - 2C_1X_1\} - \psi = 0 \qquad \text{(Eq. 4-11)}$$

$$\sum_{j=1}^{n} T_jY_j + C_1 = [b(X_t+X_1)+1] / 2b \qquad \text{(Eq. 4-14)}$$

$$\sum_{i=1}^{n} iT_i = 1 - C_0 \qquad \text{(Eq. 4-2)}$$

Substituting Eq. 4-14 into Eq. 4-11 yields:

$$X_1 + b \{X_1^2 - 2X_1[\sum_{j=1}^{n} T_jY_j+C_1]\} - \psi = 0$$

$$X_1 + b \{X_1^2 - 2X_1[b(X_t+X_1)+1]/2b\} - \psi = 0$$

$$X_1 + b \{X_1^2 - X_1X_t - X_1^2 - X_1/b\} - \psi = 0$$

$$X_1 - bX_1X_t - X_1 - \psi = 0$$

$$\psi = -bX_1X_t$$

Therefore, the three equations (Eqs. 4-11, 4-14 and 4-2) are always solvable. This completes the proof of Theorem II.

3. Locations of relative minimums for one or two payoff values

Theorem II proved that if there are at most two distinct payoff values for n states of nature, it is necessary and sufficient for a relative minimum index of utility to exist. In other words, a relative minimum does not exist if there are more than two distinct payoff values. Therefore, for the case that there are more than two distinct payoff values, the global minimum index of utility must occur at the corner boundary points.

In the case of only one payoff value, any set of T values, $(T_1, T_2, \ldots, T_n)$, satisfying $\sum_{i=1}^{n} iT_i = 1 - C_0$ and $T_i \geq 0$ is one of the multiple solutions that result in the relative minimum index of utility. Since there is only one payoff value, the maximum and minimum indexes are both equal to the payoff value.

In the case of two distinct payoff values, the relative minimum indexes can be located by solving Eqs. 4-14 and 4-2. By subtracting X_1 times Eq. 4-2 from Eq. 4-14, Eq. 4-14 becomes,

$$\sum_{i=1}^{n} T_iY_i + C_1 - X_1\left(\sum_{i=1}^{n} iT_i\right) = [b(X_t+X_1)+1]/2b - X_1(1-C_0)$$

Substituting $Y_i = a_iX_1 + (i-a_i)X_t$,

$$\sum_{i=1}^{n} [a_i X_1+(i-a_i)X_t]T_i - \sum_{i=1}^{n} iX_1 T_i = [b(X_t+X_1)+1]/2b - X_1 + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} \{[a_i X_1+(i-a_i)X_t]-iX_1\} T_i = [bX_t+bX_1+1-2bX_1]/2b + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} \{(i-a_i)X_t-(i-a_i)X_1\} T_i = [b(X_t-X_1)+1]/2b + X_1 C_0 - C_1$$

$$\sum_{i=1}^{n} (i-a_i)(X_t-X_1)T_i = [b(X_t-X_1)+1]/2b + X_1 C_0 - C_1$$

Recall that:

$$C_0 = C' + C''$$

$$C_1 = C'X_1 + C''X_t$$

$$C_2 = C'X_1^2 + C''X_t^2$$

Substituting the expression for C_0 and C_1 yields,

$$\sum_{i=1}^{n} (i-a_i)(X_t-X_1)T_i = [b(X_t-X_1)+1]/2b + X_1(C'+C'') - C'X_1 - C''X_t$$

$$\sum_{i=1}^{n} (i-a_i)(X_t-X_1)T_i = (X_t-X_1)/2 + 1/2b - (X_t-X_1)C''$$

$$\sum_{i=1}^{n} (i-a_i)T_i = 1/2 + 1/2b(X_t-X_1) - C'' \qquad (\text{since } X_t \neq X_1)$$

$$\sum_{i=1}^{t-1} (i-a_i)T_i + \sum_{i=t}^{n} (i-a_i)T_i = 1/2 + 1/2b(X_t-X_1) - C''$$

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 + 1/2b(X_t-X_1) - C'' \quad (\text{since } a_i = i \text{ for } i=1, .., t-1)$$

Eq. 4-14 is replaced by Eq. 4-15.

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' + 1/2b(X_t-X_1) \qquad \text{(Eq. 4-15)}$$

By subtracting Eq. 4-15 from Eq. 4-2, Eq. 4-2 becomes,

$$\sum_{i=1}^{n} iT_i - \sum_{i=t}^{n} (i-a_i)T_i = 1 - C_0 - 1/2 + C'' - 1/2b(X_t-X_1)$$

$$\sum_{i=1}^{t-1} iT_i + \sum_{i=t}^{n} iT_i - \sum_{i=t}^{n} iT_i + \sum_{i=t}^{n} a_iT_i = 1 - C' - C'' - 1/2 + C'' - 1/2b(X_t-X_1)$$

$$\sum_{i=1}^{t-1} iT_i + \sum_{i=t}^{n} a_iT_i = 1/2 - C' - 1/2b(X_t-X_1)$$

Since $a_i = i$ for $i = 1, 2, \ldots, t-1$, Eq. 4-2 is replaced by Eq. 4-16.

$$\sum_{i=1}^{n} a_iT_i = 1/2 - C' - 1/2b(X_t-X_1) \qquad \text{(Eq. 4-16)}$$

Eq. 4-3 remains as follows:

$$T_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \qquad \text{(Eq. 4-3)}$$

If there are only two distinct payoff values for n states of nature, any solution to Eq. 4-15 and Eq. 4-16 results in a relative minimum index. Since there are multiple solutions for Eqs. 4-15 and 4-16, multiple relative minimum indexes can be obtained from Eqs. 4-15 and 4-16. However, with the constraints of Eq. 4-3, the multiple relative minimum indexes may not be located inside the feasible region.

It is important to be able to identify conditions for which at least one relative minimum index exists inside the feasible region. Since $T_i \geq 0$, $a_i > 0$, and $(i-a_i) \geq 0$ for $i = 1, 2, \ldots, n$, the left hand side of both Eqs. 4-15 and 4-16 are greater than or equal to zero. The corresponding right hand side of these two equations must also be greater than or equal to zero. Therefore, $1/2 - C'' + 1/2b(X_t - X_1) \geq 0$ and $1/2 - C' - 1/2b(X_t - X_1) \geq 0$ are required for at least one relative minimum index of utility to exist inside the feasible region. The necessary conditions can be rewritten as:

$$- 1/2 + C'' \leq 1/2b(X_t - X_1) \leq 1/2 - C' \qquad \text{(Eq. 4-17)}$$

Despite the fact that the necessary conditions, Eq. 4-17, are met, it is still possible that none of the multiple relative minimums exists inside the feasible region. In other words, the necessary conditions indicate that there is no relative minimum inside the feasible region if they are not met. But the fulfillment of the necessary conditions does not guarantee there is at least one relative minimum inside the feasible region.

A basic solution to Eqs. 4-15 and 4-16 has at most two unknowns which are not equal to zero. There are $(n)(n-1)/2$ basic solutions to Eqs. 4-15 and 4-16. Any linear combinations of these basic solutions is also a solution to Eqs. 4-15 and 4-16. Therefore, unless all the basic solutions are outside the feasible region, at least one relative minimum occurs inside the feasible region.

4. Common value of multiple minimum indexes of utility for two payoffs

In Section E.2, it was proved that no relative minimum index of utility exists if there are more than two distinct payoff values for n ($n \geq 3$) states of nature. If there is only one payoff value, there is always only one value of the index of utility which equals the payoff value since the variance is always zero.

If there are only two distinct payoff values for n ($n \geq 3$) states of nature, multiple relative minimum indexes of utility can be located by solving Eqs. 4-15, 4-16, and 4-3.

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' + 1/2b(X_t-X_1) \qquad \text{(Eq. 4-15)}$$

$$\sum_{i=1}^{n} a_i T_i = 1/2 - C' - 1/2b(X_t-X_1) \qquad \text{(Eq. 4-16)}$$

$$T_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \qquad \text{(Eq. 4-3)}$$

Although multiple relative minimum indexes of utility exist, they result in a common value. Therefore, any solution from Eqs. 4-15, 4-16, and 4-3 is sufficient to find the common value of the multiple relative minimums.

Since there are two equations and n ($n \geq 3$) unknowns, a basic solution to Eqs. 4-15 and 4-16 has at most two unknowns which are not equal to zero. It is then reasonable to assume that only T_1 and T_t are not equal to zero while setting all other unknowns equal to zero. With the above assumption, Eqs. 4-15, 4-16 and 4-3 can be rewritten as follows,

$$(t-a_t)T_t = 1/2 - C'' + 1/2b(X_t-X_1)$$

$$a_1T_1 + a_tT_t = 1/2 - C' - 1/2b(X_t-X_1)$$

$$T_1 \geq 0, \quad T_t \geq 0$$

Since the payoff values for the states of nature from one to t-1 are all equal to X_1 according to the definition of t, then $a_t = t-1$. Therefore,

$$T_t = 1/2 - C'' + 1/2b(X_t-X_1) \qquad \text{(Eq. 4-18)}$$

$$\begin{aligned} T_1 &= 1/2 - C' - 1/2b(X_t-X_1) - (t-1)T_t \\ &= 1/2 - C' - 1/2b(X_t-X_1) - (t-1)/2 + (t-1)C'' - (t-1)/2b(X_t-X_1) \\ &= 1 - C' + (t-1)C'' - t/2 - t/2b(X_t-X_1) \qquad \text{(Eq. 4-19)} \end{aligned}$$

The expressions for T_1 and T_t in Eqs. 4-18 and 4-19 represent one of the basic solutions to Eqs. 4-15 and 4-16. This solution is feasible if both T_1 and T_t are positive. The solution is infeasible, (in other words, it is outside the feasible region) if either T_1 or T_t is negative. Whether or not the resulting values of T_1 and T_t are feasible, the common value of the minimum index of utility can always be evaluated from these values of T_1 and T_t. From Eq. 4-1,

$$I = EXP + b * VAR \qquad \text{(Eq. 4-1)}$$

$$= C_1 + \sum_{i=1}^{n} T_i Y_i + b\ \{C_2 + \sum_{i=1}^{n} T_i Z_i - C_1{}^2 - [\sum_{i=1}^{n} T_i Y_i]^2 - 2C_1[\sum_{i=1}^{n} T_i Y_i]\}$$

$$= C_1 + \sum_{i=1}^{n} T_i Y_i + b\ \{C_2 + \sum_{i=1}^{n} T_i Z_i - [C_1 + (\sum_{i=1}^{n} T_i Y_i)]^2\ \}$$

$$= C_1 + T_1Y_1 + T_tY_t + b\ \{C_2 + T_1Z_1 + T_tZ_t - [C_1 + T_1Y_1 + T_tY_t]^2\ \} \qquad \text{(Eq. 4-20)}$$

Since $Y_1 = X_1$ and $Y_t = (t-1)X_1 + X_t$, the terms $C_1 + T_1Y_1 + T_tY_t$ in Eq. 4-20 can be expressed as follows,

$$C_1 + T_1Y_1 + T_tY_t$$

$$= C_1 + T_1X_1 + T_t[(t-1)X_1+X_t]$$

$$= C'X_1 + C''X_t + T_1X_1 + T_t[(t-1)X_1+X_t] \quad \text{(since } C_1 = C'X_1 + C''X_t\text{)}$$

Substitute Eqs. 4-18 and 4-19 for T_1 and T_t,

$$C_1 + T_1Y_1 + T_tY_t$$

$$= C'X_1 + C''X_t + [1 - C' + (t-1)C'' - \frac{t}{2} - \frac{t}{2b(X_t-X_1)}]X_1$$

$$+ [\frac{1}{2} + \frac{1}{2b(X_t-X_1)} - C''] [(t-1)X_1+X_t]$$

$$= C'X_1 + C''X_t + X_1 - C'X_1 + (t-1)C''X_1 - \frac{tX_1}{2} - \frac{tX_1}{2b(X_t-X_1)}$$

$$+ \frac{(t-1)X_1}{2} + \frac{X_t}{2} + \frac{(t-1)X_1+X_t}{2b(X_t-X_1)} - C''(t-1)X_1 - C''X_t$$

$$= \frac{X_1}{2} + \frac{X_t}{2} + \frac{(t-1)X_1+X_t - tX_1}{2b(X_t-X_1)}$$

$$= \frac{X_t+X_1}{2} + \frac{(X_t-X_1)}{2b(X_t-X_1)}$$

$$= \frac{b(X_t+X_1) + 1}{2b} \qquad \text{(Eq. 4-21)}$$

Since $Z_1 = X_1^2$ and $Z_t = (t-1)X_1^2 + X_t^2$, the terms $C_2 + T_1Z_1 + T_tZ_t$ in Eq. 4-20 can be expressed as follows,

$$C_2 + T_1Z_1 + T_tZ_t$$

$$= C_2 + T_1X_1^2 + T_t[(t-1)X_1^2+X_t^2]$$

$$= C'X_1^2 + C''X_t^2 + T_1X_1^2 + T_t[(t-1)X_1^2+X_t^2] \quad \text{(since } C_2 = C'X_1^2 + C''X_t^2\text{)}$$

Substitute Eqs. 4-18 and 4-19 for T_1 and T_t,

$$C_2 + T_1Z_1 + T_tZ_t$$

$$= C'X_1^2 + C''X_t^2 + [1 - C' + (t-1)C'' - \frac{t}{2} - \frac{t}{2b(X_t-X_1)}]X_1^2$$

$$+ [\frac{1}{2} + \frac{1}{2b(X_t-X_1)} - C''] [(t-1)X_1^2+X_t^2]$$

$$= C'X_1^2 + C''X_t^2 + X_1^2 - C'X_1^2 + (t-1)C''X_1^2 - \frac{tX_1^2}{2} - \frac{tX_1^2}{2b(X_t-X_1)}$$

$$+ \frac{(t-1)X_1^2}{2} + \frac{X_t^2}{2} + \frac{(t-1)X_1^2+X_t^2}{2b(X_t-X_1)} - C''(t-1)X_1^2 - C''X_t^2$$

$$= \frac{X_1^2}{2} + \frac{X_t^2}{2} + \frac{(t-1)X_1^2+X_t^2 - tX_1^2}{2b(X_t-X_1)}$$

$$= \frac{X_t^2+X_1^2}{2} + \frac{(X_t^2-X_1^2)}{2b(X_t-X_1)}$$

$$= \frac{b(X_t^2+X_1^2) + (X_t+X_1)}{2b} \qquad \text{(Eq. 4-22)}$$

Substitute Eqs. 4-21 and 4-22 into Eq. 4-20,

$$I = C_1 + T_1Y_1 + T_tY_t + b\,\{C_2 + T_1Z_1 + T_tZ_t - [C_1 + T_1Y_1 + T_tY_t]^2\}$$

$$= \frac{b(X_t+X_1)+1}{2b} + b\,\{\frac{b(X_t^2+X_1^2)+(X_t+X_1)}{2b} - [\frac{b(X_t+X_1)+1}{2b}]^2\}$$

$$= \frac{b(X_t+X_1)+1}{2b} + b\,\{\frac{b(X_t^2+X_1^2)+(X_t+X_1)}{2b} - \frac{b^2(X_t+X_1)^2+2b(X_t+X_1)+1}{4b^2}\}$$

$$= \frac{b(X_t+X_1)+1}{2b} + b\,\{\frac{2b^2(X_t^2+X_1^2)+2b(X_t+X_1)-b^2(X_t+X_1)^2-2b(X_t+X_1)-1}{4b^2}\}$$

$$= \frac{b(X_t+X_1)+1}{2b} + \frac{2b^2X_t^2+2b^2X_1^2-b^2X_t^2-b^2X_1^2-2b^2X_tX_1-1}{4b}$$

$$= \frac{b(X_t+X_1)+1}{2b} + \frac{b^2(X_t^2+X_1^2-2X_tX_1)-1}{4b}$$

$$= \frac{b(X_t+X_1)+1}{2b} + \frac{b^2(X_t-X_1)^2 - 1}{4b}$$

$$= \frac{2b(X_t+X_1) + 2 + b^2(X_t-X_1)^2 - 1}{4b}$$

$$= \frac{b^2(X_t-X_1)^2 + 2b(X_t-X_1) + 1 + 4bX_1}{4b}$$

$$= \frac{[b(X_t - X_1) + 1]^2}{4b} + X_1 \qquad \text{(Eq. 4-23)}$$

If there are only two distinct payoff values, Eq. 4-23 is the common value of the multiple relative minimum indexes of utility. Notice that this common value becomes the global minimum index of utility only if at least one relative minimum index exists inside the feasible region.

5. Numerical example

Consider a strategy with five possible states of nature. The payoff values and the minimum differences between the states of nature are:

$X_1=10, \quad X_2=20, \quad X_3=20, \quad X_4=10, \quad X_5=10$

$k_1=.10, \quad k_2=.04, \quad k_3=.03, \quad k_4=.02, \quad k_5=.01$

And the coefficient of risk aversion is -0.25.

Solution to the numerical example: By the definition of M_i, the minimum requirements of probabilities are:

$M_1=.20, \quad M_2=.10, \quad M_3=.06, \quad M_4=.03, \quad M_5=.01$

Since there are only two distinct payoffs, the coefficients C' and C'' are:

$C' = M_1 + M_4 + M_5 = .24$ and $C'' = M_2 + M_3 = .16$

Relative minimum indexes of utility: First, check the necessary conditions.

$$-1/2 + C'' \leq 1/2b(X_t - X_1) \leq 1/2 - C' \qquad \text{(Eq. 4-17)}$$

$$-0.34 \leq -0.20 \leq 0.26$$

Since the necessary conditions are met, the relative minimum index of utility can be located by using Eqs. 4-15, 4-16, and 4-3. From Eq. 4-15,

$$\sum_{i=t}^{n} (i-a_i)T_i = 1/2 - C'' + 1/2b(X_t-X_1)$$

$$T_2 + 2T_3 + 2T_4 + 2T_5 = .5 - .16 + 1/2(-0.25)(20-10)$$

From Eq. 4-16,

$$\sum_{i=1}^{n} a_i T_i = 1/2 - C' - 1/2b(X_t-X_1)$$

$$T_1 + T_2 + T_3 + 2T_4 + 3T_5 = .5 - .24 - 1/2(-0.25)(20-10)$$

Solving these two equations simultaneously, one of the basic solutions is:

$$T_1 = .32, \quad T_2 = .14, \quad T_3 = T_4 = T_5 = 0$$

Since this is a feasible solution, the resulting relative minimum is a global minimum index of utility. And the value of the global minimum index of utility can be obtained by Eq. 4-23,

$$I = \frac{[b(X_t-X_1) + 1]^2}{4b} + X_1$$

$$= \frac{[(-0.25)(20-10) + 1]^2}{4(-0.25)} + 10$$

$$= 7.75$$

Indexes of utility at corner boundary points: The five corner boundary points are $T_i = (1-C_0)/i$ (i = 1, 2, 3, 4, or 5) with all other T_j ($j \neq i$) set at zero. The values of the mean, variance, and index at the corner solutions can be evaluated and expressed in the following table,

Corner Boundary Point	Resulting Probabilities P_1	P_2	P_3	P_4	P_5	Mean Value	Variance Value	Index of Utility
1	.80	.10	.06	.03	.01	11.60	13.44	8.24
2	.50	.40	.06	.03	.01	14.60	24.84	8.39
3	.40	.30	.26	.03	.01	15.60	24.64	9.44
4	.35	.25	.21	.18	.01	14.60	24.84	8.39
5	.32	.22	.18	.15	.13	14.00	24.00	8.00

The resulting global minimum index of utility, 7.75, lies outside the range 8.00 to 9.44 defined by the corner boundary points. The global maximum index of utility is 9.44.

6. Summary of solutions for n dimensional problem

In summary, the assessment of the extreme indexes of utility under strict ranking in the context of incomplete knowledge must proceed in the following way.

a. Global maximum At least one global maximum occurs at the corner boundary points despite the number of distinct payoff values.

b. Global minimum.

  i. If there are more than two distinct payoff values, the global minimum index of utility must occur at corner boundary points.

  ii. If there are only two distinct payoff values, it is possible that relative minimums exist inside the feasible region. Check the necessary conditions using Eq. 4-17. If the necessary conditions are not met, the global minimum will still occur at corner boundary points. If the necessary conditions are met, Eqs. 4-15, 4-16, and 4-3 are served to locate the relative minimums. Check all basic solutions to Eqs. 4-15 and 4-16.

Case 1: If at least one basic solution of Eqs. 4-15 and 4-16 is feasible, the global minimum index of utility can be calculated by Eq. 4-23.

Case 2: If all the basic solutions to Eqs. 4-15 and 4-16 are not feasible, the global minimum index of utility must occur at one of the corner boundary points.

V. DECISION PROCEDURE IN THE CONTEXT OF UNCERTAINTY AND RISK

A. Nature of the Coefficient of Risk Aversion

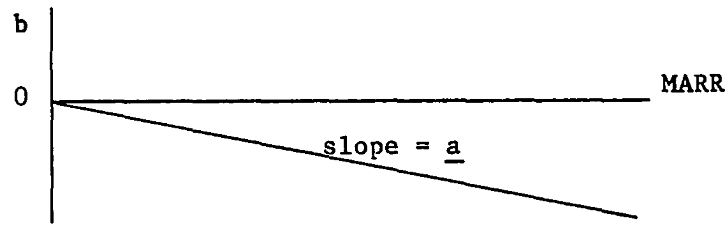
Kmietowicz and Pearman [1981] incorporated the expected value and the variance into a single index, index of utility, by introducing the coefficient of risk aversion under weak ranking in the context of incomplete knowledge. However, they did not provide a methodology to determine the value of the coefficient of risk aversion. Constant [1983] developed an optimal decision line in the context of uncertainty for one set of mutually exclusive alternatives. In the decision procedure, Constant related the coefficient of risk aversion to the minimum attractive rate of return (MARR) by an angular coefficient. The central theme of Constant's decision procedure was to ascertain the appropriate value of the angular coefficient.

Constant assumed a linear relationship between the coefficient of risk aversion and MARR which can be expressed as follows,

$$b = \underline{a} * m \qquad \text{(Eq. 5-1)}$$

where b = coefficient of risk aversion
$\underline{a}$ = angular coefficient
$\overline{m}$ = minimum attractive rate of return

Recall that the coefficient of risk aversion is a trade-off between the expected value and the variance of payoff values. A negative coefficient of risk aversion denotes an aversion to risk; a positive coefficient of risk aversion denotes a preference to risk. A negative b value therefore indicates an attempt to avoid risk by imposing a penalty on the variance. As the value of b becomes increasingly negative, the more aversive to risk (conservative) the decision maker becomes. Hence as MARR increases, the coefficient of risk aversion, b, becomes more negative at a constant rate. This constant rate is defined as the angular coefficient with the symbol "a".



In Constant's decision procedure, both positive and negative a values were considered as possible candidates for the final decision line. However, careful study of the relationship between the coefficient of risk aversion and MARR denies any consideration of positive a values based on the following logic:

> If a is positive, the value of b becomes more positive (i.e., more risk preferable) as the value of the minimum attractive rate of return becomes more positive (i.e., more risk aversive). This contrary statement dictates that the assumption of a positive a value cannot be true.

Therefore, since all positive a values should be discarded, the upper limit of a values must be zero.

B. Constant's Decision Procedure in the Context of Uncertainty

By assuming equal probabilities for all possible states of nature (Bayes-Laplace criterion), Constant [1983] developed the decision procedure for the context of uncertainty. Steps in this decision procedure were as follows:

1. For each state of nature, solve for the rate of return comparing each alternative with present conditions and with one another.

2. Through the use of network diagrams, form a decision line for each state of nature.

3. From the decision lines for all possible states of nature, develop the line of dominance. The line of dominance finds the ranges of MARR for which the strategies are always optimal. Between the dominant ranges lie the indeterminable regions.

4. For each pair of alternatives, i.e., alternative compared with present conditions or compared with one another, draw a line of indiscernibility between the highest rate of return and the lowest rate of return representing the "best case" and "worst case" scenarios.

5. Using Bayes-Laplace criterion, calculate the expected value and variance of the payoffs (AEX or PEX) obtained from all possible states of nature for each pair of alternatives.

   a) Calculate AEX or PEX for each state of nature at an initial trial rate of return.

   b) Calculate the expected value and variance for the AEX or PEX value of each strategy.

6. Combine the expected value and variance into a single index, index of utility.

$$I = EXP[AEX] + b * VAR[AEX]$$

$$= EXP[AEX] + \frac{a}{m} * VAR[AEX] \qquad \text{(Eq. 5-2)}$$

where I = index of utility to the decision maker
EXP[AEX] = expected value of the criteria, i.e., AEX
VAR[AEX] = variance of the criteria
b = coefficient of risk aversion
a = angular coefficient
m = trial rate of return

7. Starting with an arbitrary $\frac{a}{m}$ value, by trial and error find the utility index rate of return for each alternative compared over present conditions and with one another. This is the rate of return that causes the index of utility to equal zero.

$I = EXP[AEX] + \underline{a} * m * VAR[AEX] = 0$

For each trial rate of return, the following will result:

a) AEX for all possible states of nature
b) EXP[AEX] and VAR[AEX]
c) Index of Utility, I.

Repeat this step for a range of $\underline{a}$ values.

8. Through the use of network diagrams draw a decision line based on the utility index ROR for each $\underline{a}$ value used in Step 7.

9. Plot a curve showing the utility index ROR against the $\underline{a}$ values for each pair of alternatives on the decision lines of utility index ROR.

10. On each curve of the utility index ROR respect to the $\underline{a}$ values, darken the section corresponding to the indiscernible region obtained in Step 4. The extreme points of the indiscernible region represent the "worst case" scenario (lowest rate of return) and the "best case" scenario (highest rate of return). Constant refers to the corresponding $\underline{a}$ value for the "worst case" scenario (lowest rate of return) as $\underline{a}_{min}$. This value is the lower limit for possible $\underline{a}$ values for each pair of alternatives.

11. By inspection of the $\underline{a}_{min}$ values for each pair of alternatives, choose the maximum (least negative) of the $\underline{a}_{min}$ values. This value is referred to as $max(\underline{a}_{min})$. This value is used to calculate the final decision line. It is chosen because it represents the lower limit of a common range of $\underline{a}$ values.

```
---------------------------------+=================+
                               a_min               0

--------------------------------------+============+
                                    a_min **       0

----------------------------+======================+
                          a_min                    0

------------------------------+====================+
                            a_min                  0

<------more conservative     less conservative------>
```

The starred $\underline{a}_{min}$ value is $max(\underline{a}_{min})$. A final decision line based on this $\underline{a}$ value gives the decision maker the most conservative guideline. This final decision line leads to the selection of alternatives that will result in a minimum loss even if the worst state of nature occurs. As the $\underline{a}$ values become less negative, the decision maker's choices become less conservative.

12. Recalculate the final decision line based on the max($\underline{a}_{min}$) value.

Constant also described $\underline{a}_{max}$ values. The $\underline{a}_{max}$ values were defined as the corresponding $\underline{a}$ value for the "best case" scenarios (highest rate of return for the indiscernible region). The $\underline{a}_{max}$ values are always positive which is meaningless as discussed in Section A. Hence, $\underline{a}_{max}$ values will not be recognized in this research.

Apart from deviating from the incorporation of $\underline{a}_{max}$ values into the analysis, the logic used by Constant is quite correct:

1. The monotonically decreasing feature of the curves (utility index ROR in respect to the angular coefficient) dictates a one-to-one correspondence between the utility index ROR and the $\underline{a}$ values for each pair of alternatives. Therefore, one and only one corresponding $\underline{a}$ value can be found for any given rate of return.
2. The use of the index of utility, considering both the expected value and variance which are traded off by the coefficient of risk aversion, enables the analyst to evaluate all alternatives by a single criterion.
3. The max($\underline{a}_{min}$) value found by Constant causes the final decision line to ignore a strategy if the minimum attractive rate of return (MARR) is set higher than the rate of return of the "worst case" scenarios. In other words, the final decision line is extremely risk aversive toward the initial investments (alternatives compared to present conditions). As the MARR value becomes lower, the final decision line is less risk aversive. As

a result, incremental investments could be selected by the final decision line. The incremental investments may result in either a higher or a lower rate of return than MARR. However, the incremental investments will still result in a rate of return higher than MARR in the long run.

C. Application of Constant's Decision Procedure

Constant's technique is best illustrated by presentation of a problem. Let us consider the comparison of four mutually exclusive energy conservation measures, A, B, C, and D, being compared over present conditions, P, as well as with one another. Data for the four projects are as follows:

	Cost of Investment ($)	Cost of Electricity ($/yr)	Cost of Natural Gas ($/yr)	Life of Project (yrs)
P	-	8,000.00	6,000.00	-
A	10,000.00	7,000.00	3,500.00	5
B	14,000.00	3,200.00	6,000.00	5
C	16,000.00	8,000.00	600.00	5
D	20,000.00	3,450.00	4,000.00	5

Let there be four possible states of nature, N_1, N_2, N_3, and N_4 as follows:

	N_1	N_2	N_3	N_4
Escalation rate of electricity	0%	+2%	+2%	+1%
Escalation rate of natural gas	0%	+1%	-1%	+3%

The following steps are necessary for Constant's procedure:

1. For each state of nature, the rates of return were calculated comparing each project with present conditions and with each other. For state of nature N_1, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	22.11%			
B	21.15%	18.72%		
C	20.42%	17.57%	15.24%	
D	19.06%	15.94%	14.05%	13.46%

For state of nature N_2, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	23.68%			
B	23.57%	23.30%		
C	21.63%	18.12%	5.91%	
D	21.08%	18.45%	15.03%	18.93%

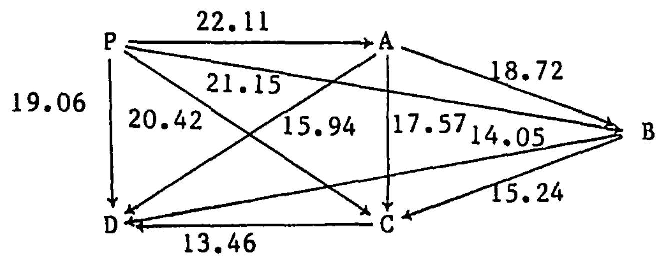
For state of nature N_3, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	21.96%			
B	23.57%	27.33%		
C	19.22%	14.43%	-48.22%	
D	20.38%	18.81%	12.40%	24.61%

For state of nature N_4, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	25.08%			
B	22.36%	14.99%		
C	24.04%	22.29%	34.26%	
D	20.99%	16.71%	17.79%	7.11%

2. Through the use of network diagrams form a decision line for each state of nature. For state of nature N_1, the network diagram is as follows:



If these estimates for N_1 were single-valued estimates, the following conclusion would be drawn:

If MARR is greater than 22.11%, choose present conditions.
If MARR is between 18.72% and 22.11%, choose alternative A.
If MARR is between 15.24% and 18.72%, choose alternative B.
If MARR is between 13.46% and 15.24%, choose alternative C.
If MARR is less or equal to 13.46%, choose alternative D.

This logic was used to establish a decision line for each possible state of nature as shown as follows. The four decision lines facilitate the search for the zones of dominance resulting in the line of dominance.

N_1:

```
               D                    C   B      A       P
+-----------------------------------+---+------+-------+-----------
0                              13.46% 15.24% 18.72% 22.11%
```

N_2:

```
                D                        B            A  P
+------------------------------------+----------------++------
0                                 15.03%          23.30% 23.68%
```

N_3:

```
               D                             B                  P
+----------------------------+-------------------------+-------
0                         12.40%                    23.57%
```

N_4:

```
          D                       C                          A  P
+---------------+---------------------------------------+----+---
0             7.11%                                22.29%  25.08%
```

3. From the decision lines for all possible states of nature, develop the line of dominance as follows:

```
          D                  indeterminable region                P
+===============+---------------------------------------------+====
0             7.11%                                       25.08%
```

4. For each pair of alternatives, i.e., alternative compared with present conditions or compared with one another, draw a line of indiscernibility between the highest rate of return and the lowest rate of return representing the "best case" and "worst case" scenarios, respectively.

For the ten pairs of alternatives, i.e., 1) A over P; 2) B over P; 3) B over A; 4) C over P; 5) C over A; 6) C over B; 7) D over P; 8) D over A; 9) D over B; and 10) D over C. Considering A over P, state of nature N_1 gives the lowest rate of return, 21.96%; state of nature N_4 gives the highest rate of return, 25.08%. Therefore, the indiscernible region for A over P is between 21.96% and 25.08%. The indiscernible regions for all pairs are as follows:

```
A - P +-------------------------------------------******-----------
                                               21.96%  25.08%

B - P +-----------------------------------------*****--------------
                                             21.15% 23.57%

B - A +-----------------------------***********************--------
                                  14.99%                27.33%

C - P +-------------------------------------**********-------------
                                          19.22%   24.04%

C - A +----------------------------****************----------------
                                 14.43%         22.29%

C - B *************************************************************
      0                                                      34.26%

D - P +-------------------------------------*****------------------
                                         19.06%  21.08%

D - A +--------------------------------******----------------------
                                   15.94%   18.81%

D - B +------------------------************------------------------
                             12.40%     17.99%

D - C +-----*******************************************------------
           7.11%                                    24.61%
```

5. Using Bayes-Laplace criterion, calculate the expected value and variance of AEX obtained from all possible states of nature for each pair of alternatives.

   a) Calculate AEX for each state of nature at an initial trial rate of return,

      where:

$$AEX = \left\{ \sum_{j=1}^{5} [FS_1*(1+e_{F1})^j + FS_2*(1+e_{F2})^j] * (p/f)^j_{i_a} - B \right\} * (a/p)^5_{i_a}$$

For alternative A compared to present conditions, the AEX values for the various states of nature using an initial trial rate of return of 20% were as follows:

N_1: e_{F1} = 0%, e_{F2} = 0%, AEX_1 = 156.20
N_2: e_{F1} = 2%, e_{F2} = 1%, AEX_2 = 277.08
N_3: e_{F1} = 2%, e_{F2} =-1%, AEX_3 = 145.04
N_4: e_{F1} = 1%, e_{F2} = 3%, AEX_4 = 388.14

b) Calculate the expected value and variance for the AEX value for each pair of alternatives.

$$EXP[AEX] = \sum_{j=1}^{4} P_j * AEX_j$$

$$VAR[AEX] = \sum_{j=1}^{4} P_j * (AEX_j)^2 - (EXP[AEX])^2$$

Since by Bayes-Laplace assumption, $P_1 = P_2 = P_3 = P_4 =$ 25%, the expected value and variance for alternative A compared with present conditions were as follows:

EXP[AEX] = 241.62

VAR[AEX] = 9,837.08

6. Combine the expected value and the variance into an index of utility.

$$I = EXP[AEX] + \underline{a} * m * VAR[AEX]$$

where $\underline{a}$ = angular coefficient
$\underline{m}$ = trial rate of return

Continuing the example illustrated above (alternative A over present conditions) and assuming the angular coefficient is set at -0.02 and the trial rate of return at 20%, the index of utility is calculated as follows:

$$I = 241.62 + (-0.02)(20\%)(9837.08) = 202.27$$

7. Starting with an arbitrary $\underline{a}$ value, find the utility index ROR for each alternative compared over present conditions and with one another. This is the rate of return that causes the utility function to equal zero.

If the $\underline{a}$ value is set at -0.02, the utility index ROR of 22.65%

for alternative A compared with present conditions is found by trial and error as follows:

Setting i equal to 22.65%,

N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = -40.88$
N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 78.05$
N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = -51.90$
N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 187.27$

By Bayes-Laplace assumption, $P_1 = P_2 = P_3 = P_4 = 25\%$, the expected value and variance were as follows:

$$EXP[AEX] = 43.14$$

$$VAR[AEX] = 9,521.34$$

The index of utility thus reduces to zero:

$$I = 43.14 + (-0.02)(22.65\%)(9521.34) = 0$$

For $\underline{a} = -0.02$, the utility index ROR for the other pairs of alternatives compared over present conditions and each project compared to one another are summarized as follows:

	Internal ROR on Utility function over P	A	B	C
Alternative A	22.65%			
B	22.20%	18.75%		
C	19.80%	16.98%	5.12%	
D	20.00%	17.14%	14.38%	13.14%

Step 7 is repeated for a range of negative $\underline{a}$ values, i.e., 0, -0.02, -0.04, -0.06, -0.08, -0.10, -0.12, -0.14, -0.16, and -0.20.

8. Through the use of network diagrams draw a decision line based on the utility index ROR for each $\underline{a}$ value used in Step 7.

$\underline{a} = 0$,

```
                    D                              B            A     P
+--------------------------------------------+-------------+----+---
0                                         14.86%        21.30%  23.22%
```

$\underline{a} = -0.02$,

```
                  D                           B            A        P
+-------------------------------------+-----------+-----------+------
0                                  14.38%      18.75%      22.65%
```

a = -0.04,

```
                        D                                 A           P
+--------------------------------------------------------+-----------+-----
0                                                   16.79%       22.10%
```

a = -0.06,

```
                        D                                  A            P
+-----------------------------------------------------+-----------+-------
0                                                16.46%      21.57%
```

a = -0.08,

```
                         D                                A             P
+----------------------------------------------------+-----------+--------
0                                               16.14%      21.05%
```

a = -0.10,

```
                         D                                A             P
+--------------------------------------------------+-----------+----------
0                                             15.82%      20.55%
```

a = -0.12,

```
                         D                               A              P
+-------------------------------------------------+------------+----------
0                                            15.52%      20.06%
```

a = -0.14,

```
                 D                       B                              P
+-------------------------+-----------------------------+-----------------
0                    11.96%                       19.62%
```

a = -0.16,

```
                 D                       B                              P
+------------------------+-----------------------------+------------------
0                   11.61%                       19.23%
```

a = -0.20,

```
                 D                       B                              P
+----------------------+-----------------------------+--------------------
0                 10.98%                       18.48%
```

Note that only five pairs of viable alternatives, i.e., A over P, B over P, B over A, D over A, and D over B appear on these decision lines of utility index ROR. These pairs completely dominate the other pairs of alternatives. Subsequently, only these pairs are carried for further discussion.

9. Plot a curve showing the utility index ROR against the a values for each pair of alternatives that appeared on the decision lines of utility. See Figure 1.

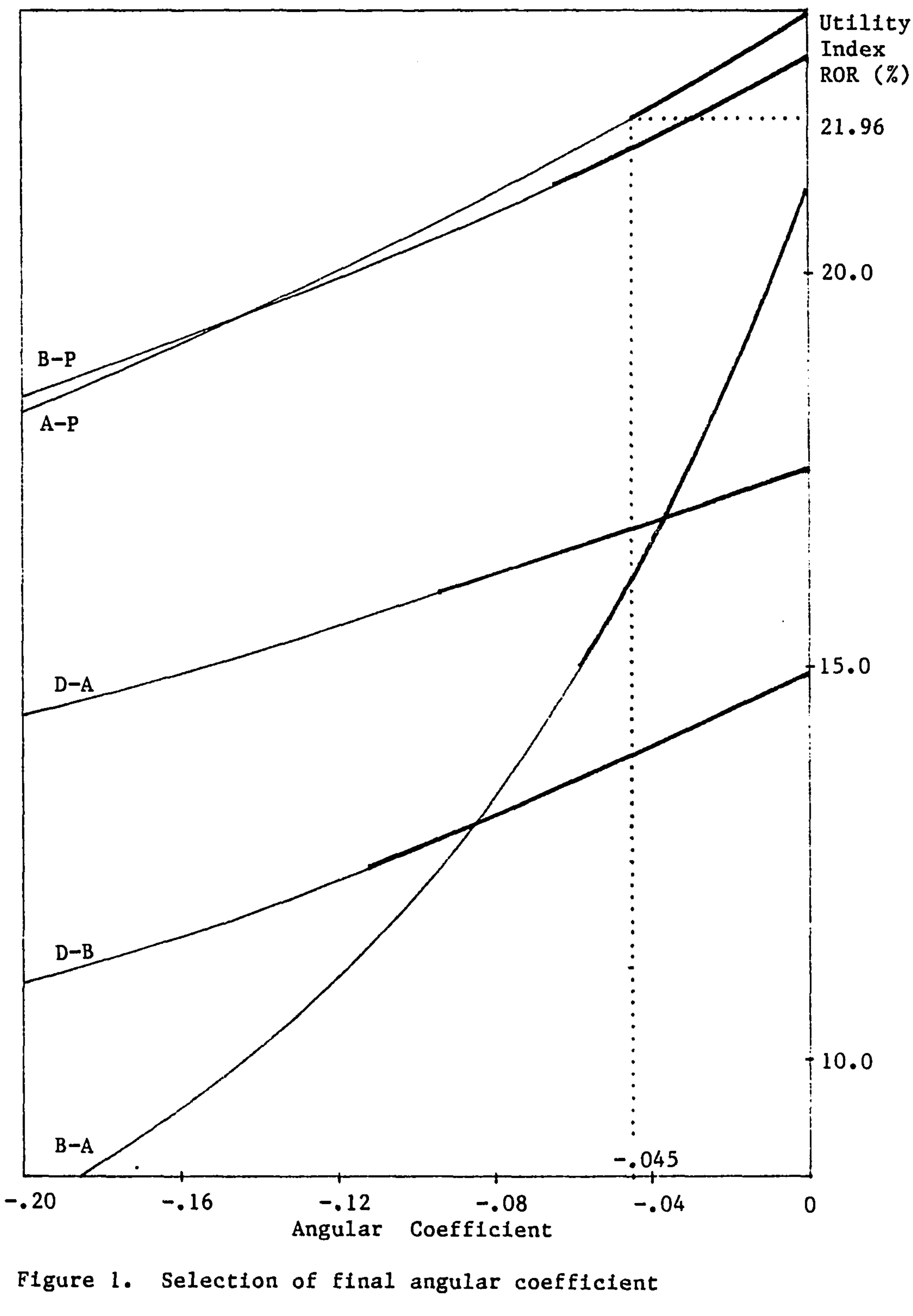


Figure 1. Selection of final angular coefficient

10. Darken the section on each curve that corresponds to the indiscernible regions obtained in Step 4. See Figure 1. For example, for curve A over P, the extreme point is 21.96% ("worst case" scenario). The corresponding $\underline{a}_{min}$ value is -0.045 for the "worst case" scenario.

    For the remaining curves, the $\underline{a}_{min}$ values corresponding to the "worst case" scenarios are as follows:

Pair of alternatives	Corresponding $\underline{a}_{min}$
A over P	-0.045 **
B over P	-0.066
B over A	-0.058
D over A	-0.092
D over B	-0.115

11. By inspection of the $\underline{a}_{min}$ values for the viable alternatives, the maximum of the $\underline{a}_{min}$ values is -0.045.

12. Recalculate the final decision line based on the max($\underline{a}_{min}$) value of -0.045 as described in Step 7. The final decision line of utility index ROR is as follows:

```
                  D                                A           P
+------------------------------------------+------------+---------
0                                        16.71%       21.96%
```

For the four possible states of nature in this problem, the decision maker will choose the alternative based on the following reasoning:

If MARR is greater that 21.96%, choose present conditions.
If MARR is between 16.71% and 21.96%, choose project A.
If MARR is less that 16.71%, choose project D.

Referring back to the line of dominance in Step 3, note that the cutoff point for projects to be initially accepted over present conditions has become more stringent. This phenomenon reflects the fact that the most conservative $\underline{a}$ value was chosen to formulate the final decision line.

Most conservatively, the decision does nothing if MARR is greater or equal to 21.96% even it is likely to realize a rate of return higher than 21.96%. The range of MARR values for which it is economically feasible to choose project A has been determined. The indeterminable region between project D and project A has now been clarified. Acceptance of project D becomes less stringent reflecting the fact that the decision maker is willing to take some risks as the MARR value becomes lower. Project D can now be accepted if MARR is less than 16.71% versus 7.11%. The fact that all possible states of nature have been considered lead to a higher probability of accepting project D.

D. Simplified Approach for Constant's Decision Procedure

Although the decision procedure developed by Constant is correct theoretically, it is very tedious and time consuming to apply. Therefore, Constant's approach for determining the final decision line under the context of uncertainty has been simplified by this author.

The first, and fourth steps (Steps 2 and 3 are eliminated) of the simplified approach are the same as in the Constant's approach. Addressing the lines of indiscernibility developed in Step 4, let the lowest rate of return ("worst case" scenario) be defined as i_{min}, and the highest rate of return ("best case" scenario) as i_{max}.

Because of the one-to-one correspondence between the utility index ROR and the angular coefficients, Figure 1, there is one and only one angular coefficient correspondent to each utility index ROR for any pair of alternatives. Also, the index of utility is equal to zero for each utility index ROR at the corresponding angular coefficient. The equation for the index of utility, Eq. 5-3, can be rewritten as follows:

$$I = 0 = EXP[AEX] + b * VAR[AEX]$$

$$= EXP[AEX] + \underline{a} * m * VAR[AEX]$$

Therefore, the corresponding angular coefficient can be found by the following equation:

$$\underline{a} = \frac{-EXP[AEX]}{(m)(VAR[AEX])} \qquad \text{(Eq. 5-3)}$$

The possible range of angular coefficients corresponding to the line of indiscernibility, with upper limit of $\underline{a}_{max}$ and lower limit of $\underline{a}_{min}$, can now be calculated by directly using the following equations:

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 5-4)}$$

$$\underline{a}_{j,max} = \frac{-\ EXP[AEX]_j}{(i_{j,max})(VAR[AEX]_j)} \qquad \text{(Eq. 5-5)}$$

where: j = a certain pair of alternatives
$\underline{a}_{j,min}$, $\underline{a}_{j,max}$ = lower and upper limit of angular coefficient, respectively
$i_{j,min}$, $i_{j,max}$ = lowest and highest rates of return on the line of indiscernibility, respectively
$EXP[AEX]_j$ = expected value of payoffs based on $i_{j,min}$ or $i_{j,max}$
$VAR[AEX]_j$ = variance of payoffs based on $i_{j,min}$ or $i_{j,max}$

Because i_{min} is the smallest rate of return among all possible states of nature for any pair of alternatives, the payoffs (AEX or PEX) for the various states of nature must all be greater than or equal to zero. Therefore, the expected value of the payoffs, EXP[AEX], will always be positive based on i_{min}. By the same reasoning, the expected value of the payoffs will always be negative based on i_{max}. Taking alternative A over P as an example, the values of AEX based on both i_{min} and i_{max} are as follows:

Line of indiscernibility:

```
A - P +-------------------------------------------******------------
                                            21.96%  25.08%
```

State of Nature	e_{F1}	e_{F2}	AEX based on $i_{min} = 21.96\%$	AEX based on $i_{max} = 25.08\%$
N_1	0%	0%	11.05	-224.80
N_2	2%	1%	130.49	-107.59
N_3	2%	-1%	0.00	-235.69
N_4	1%	3%	240.19	0.00

Since the variance will always be positive by definition, the above equations dictate that the value of $\underline{a}_{min}$ is negative if i_{min} is positive; and the value of $\underline{a}_{min}$ is positive if i_{min} is negative. The value of $\underline{a}_{max}$ is positive if i_{max} is positive; and the value of $\underline{a}_{min}$ is negative if i_{min} is negative.

The induction above reveals two difficulties in searching for the possible range of the angular coefficients for each pair of alternatives. First, a positive angular coefficient is meaningless, as discussed in Section A. Therefore, the possible value of the angular coefficient must be limited to a negative value or zero in all cases.

Second, a negative rate of return is meaningless in economic decision process. Therefore, the values for the rates of return must be limited to positive values or zero.

Since $\underline{a}_{max}$ is always positive for positive i_{max}, zero will always substitute for positive values of $\underline{a}_{max}$. Hence, there is no need to calculate $\underline{a}_{max}$ in searching for the possible range of angular coefficients. From this point only $\underline{a}_{min}$ will be considered.

The next step is to find the maximum from among the $\underline{a}_{min}$ values for each pair of alternatives. The range of possible angular coefficients for the final decision line is then defined by the the value of max($\underline{a}_{min}$) and zero. If the decision maker is very conservative, the lower limit of this possible range of $\underline{a}$ values, i.e., the max($\underline{a}_{min}$), will be chosen as the angular coefficient to calculate the final decision line.

In summary, the necessary steps in the simplified approach are restated as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

2. For all pairs of alternatives, i.e., alternatives compared with present conditions and with one another, draw a line of indiscernibility between the lowest rate of return and the highest rate of return representing the "best case" and "worst case" scenarios. Let the lowest rate of return ("worst case" scenario) be defined as i_{min}, and the highest rate of return ("best case" scenario) as i_{max}.

3. Calculate the lower limits of possible angular coefficients for all pairs of alternatives from the lines of indiscernibility according to the following equations:

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 5-4)}$$

4. For all pairs of alternatives, take the largest (least negative) value of $\underline{a}_{j,min}$. The range for possible final $\underline{a}$ values is then defined by the maximum of $\underline{a}_{min}$ (which is a negative value) and zero. The max($\underline{a}_{min}$) value is the most conservative angular coefficient within the possible range; as the angular coefficients within the range becomes less negative, decisions become less conservative.

5. Calculate the final decision line of utility index ROR based on the angular coefficient selected using the following equation in a trial and error routine.

$$I_j = 0 = EXP[AEX]_j + \underline{a} * r_j * VAR[AEX]_j$$

where I_j = index of utility for the pair of alternatives j
$\underline{a}$ = final selected angular coefficient
$\overline{r}_j$ = rate of return for the pair of alternatives j

6. Examine the final decision line of utility index ROR. If the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is included on the final decision line, the decision procedure is completed. However, if the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is not included on the final decision line, the final $\underline{a}_{min}$ value must be modified. Choose the next least negative value of $\underline{a}_{min}$ as the revised final angular coefficient. Steps 5 and 6 are repeated as many times as necessary.

The same numerical example used for Constant's approach will be presented again to illustrate the simplified approach. The necessary steps are as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

For state of nature N_1, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	22.11%			
B	21.15%	18.72%		
C	20.42%	17.57%	15.24%	
D	19.06%	15.94%	14.05%	13.46%

For state of nature N_2, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	23.68%			
B	23.57%	23.30%		
C	21.63%	18.12%	5.91%	
D	21.08%	18.45%	15.03%	18.93%

For state of nature N_3, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	21.96%			
B	23.57%	27.33%		
C	19.22%	14.43%	-48.22%	
D	20.38%	18.81%	12.40%	24.61%

For state of nature N_4, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	25.08%			
B	22.36%	14.99%		
C	24.04%	22.29%	34.26%	
D	20.99%	16.71%	17.79%	7.11%

2. For all pairs of alternatives, i.e., alternative compared with present conditions and with one another, draw a line of indiscernibility between the highest rate of return and the lowest rate of return representing the "best case" and "worst case" scenarios, respectively. Let the highest rate of return be defined as i_{max}, and the lowest rate of return as i_{min}. The lines of indiscernibility for all pairs of alternatives on the decision line are as follows:

```
A - P +--------------------------------------------******-----------
                                                 21.96% 25.08%

B - P +------------------------------------------*****--------------
                                               21.15% 23.57%

B - A +-----------------------------**************************--------
                                   14.99%                  27.33%

C - P +---------------------------------------**********-------------
                                           19.22%   24.04%

C - A +-----------------------------****************----------------
                                  14.43%          22.29%

C - B **************************************************************
       0                                                      34.26%

D - P +--------------------------------------*****------------------
                                         19.06%  21.08%

D - A +-------------------------------*****-------------------------
                                   15.94%  18.81%

D - B +-------------------------************-------------------------
                              12.40%     17.99%

D - C +-----*********************************************-----------
          7.11%                                        24.61%
```

3. Calculate the lower limits of possible angular coefficients from the lines of indiscernibility according to Eq. 5-4:

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 5-4)}$$

For alternative A compared with present conditions, the value of i_{min} is found to be 21.96%. The following payoffs (AEX in this case) can be calculated based on 21.96%:

N_1: e_{F1} = 0%, e_{F2} = 0%, AEX_1 = 11.05
N_2: e_{F1} = 2%, e_{F2} = 1%, AEX_2 = 130.49
N_3: e_{F1} = 2%, e_{F2} =-1%, AEX_3 = 0.00
N_4: e_{F1} = 1%, e_{F2} = 3%, AEX_4 = 240.19

By Bayes-Laplace assumption, $P_1 = P_2 = P_3 = P_4$ = 25%, the expected value and variance were as follows:

EXP[AEX] = 95.43

VAR[AEX] = 9,602.47

The value of $\underline{a}_{min}$ for alternative A over P is then calculated as follows:

$$\underline{a}_{min} = \frac{-EXP[AEX]}{(i_{min})(VAR[AEX])} = \frac{-95.43}{(21.96\%)(9{,}602.47)} = -0.045260$$

The other values of $\underline{a}_{min}$ from the other lines of indiscernibility are calculated in the same manner and listed as follows:

Pair of alternatives	$\underline{a}_{min}$ values
A - P	-0.045260
B - P	-0.066416
B - A	-0.058306
C - P	-0.027929**
C - A	-0.070396
C - B	+0.001713
D - P	-0.071067
D - A	-0.092246
D - B	-0.114836
D - C	-0.093491

4. From among all pairs of alternatives, choose the largest value that is negative (i.e., least negative) of $\underline{a}_{j,min}$, -0.027929 (C over P). This value is the most conservative angular coefficient within the possible range from -0.027929 to zero.

5. Calculate the final decision line of utility index ROR based on the angular coefficient selected, i.e., -0.027929.

```
                    D                       B          A          P
+-----------------------------------------+----------+----------+---------
0                                     14.20%     17.86%     22.43%
```

6. Since the final decision line does not include the pair of alternatives C over P, from which the final angular coefficient is selected, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.045260 (A over P).

5a. Calculate the final decision line based on the angular coefficient selected, i.e., -0.045260.

```
                    D                                  A             P
+----------------------------------------------------+-------------+---------
0                                                16.71%        21.96%
```

6a. Since the final decision line does include the pair of alternatives A over P, from which the final angular coefficient is selected, the final decision line is completed. This final decision line is identical to that obtained in Constant's approach.

E. Application of Simplified Approach to the Context of Risk

The simplified approach is readily applied to the context of risk with a minor modification. The modification is that the calculation of the expected value and variance is according to the exact probabilities predicted by the decision maker, instead of Bayes-Laplace assumption. Assuming that the decision maker predicts a probability of 40% for state of nature N_1, 30% for N_2, 20% for N_3, and 10% for N_4, the simplified approach for the same numerical example can be applied as follows:

1-2. Same as Steps 1 to 2 in Section D.

3. Calculate the lower limits of possible angular coefficients from the lines of indiscernibility according to Eq. 5-4:

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 5-4)}$$

For alternative A compared with present conditions, the value of i_{min} is found to be 21.96%. The following payoffs (AEX in this case) can be calculated based on 21.96%:

N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = 11.05$
N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 130.49$
N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = 0.00$
N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 240.19$

Since $P_1 = 40\%$; $P_2 = 30\%$; $P_3 = 20\%$; and $P_4 = 10\%$, the expected value and variance were calculated as follows:

$$\begin{aligned} EXP[AEX] &= (.40)11.05+(.30)130.49+(.20)0.00+(.10)240.19 \\ &= 67.59 \end{aligned}$$

$$\begin{aligned} VAR[AEX] &= (.40)(11.05)^2+(.30)(130.49)^2+(.20)(0.00)^2 \\ &\quad +(.10)(240.19)^2 - (67.59)^2 \\ &= 6{,}358.15 \end{aligned}$$

The value of $\underline{a}_{min}$ for alternative A over P is then calculated as follows:

$$\underline{a}_{min} = \frac{-EXP[AEX]}{(i_{min})(VAR[AEX])} = \frac{-67.59}{(21.96\%)(6{,}358.15)} = -0.048411$$

The other values of $\underline{a}_{min}$ from the other lines of indiscernibility are calculated in the same manner and listed as follows:

Pair of alternatives	$\underline{a}_{min}$ values
A - P	-0.048411
B - P	-0.045292
B - A	-0.086047
C - P	-0.039390**
C - A	-0.110434
C - B	+
D - P	-0.045854
D - A	-0.073886
D - B	-0.170803
D - C	-0.146645

4. From among all pairs of alternatives, choose the least negative value of $\underline{a}_{j,min}$, -0.039390 (C over P). This value is the most conservative angular coefficient within the possible range from -0.039390 to zero.

5. Calculate the final decision line based on the angular coefficient selected, i.e., -0.039390.

```
                    D                       B           A        P
+---------------------------------------+-----------+--------+--------
                                     13.91%      18.12%   22.12%
```

6. Since the final decision line does not include the pair of alternatives C over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.045292 (B over P).

5a. Calculate the final decision line based on the angular coefficient selected, i.e., -0.045292.

```
                    D                      B            A        P
+---------------------------------------+-----------+--------+--------
0                                    13.83%      17.67%   22.01%
```

6a. Since the final decision line does not include the pair of alternatives B over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.045854 (D over P).

5b. Calculate the final decision line based on the angular coefficient selected, i.e., -0.045854.

```
                    D                      B           A         P
+---------------------------------------+----------+----------+--------
0                                    13.83%      17.63%    22.00%
```

6b. Since the final decision line does not include the pair of alternatives D over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.048411 (A over P).

5c. Calculate the final decision line based on the angular coefficient selected, i.e., -0.048411.

```
                    D                     B           A          P
+---------------------------------------+---------+-----------+--------
0                                    13.80%    17.44%      21.96%
```

6c. Since the final decision line does include the pair of alternatives A over P, the final decision line is completed.

VI. DECISION PROCEDURE FOR WEAK RANKING IN THE CONTEXT OF INCOMPLETE KNOWLEDGE

For weak ranking in the context of incomplete knowledge, Cannon and Kmietowicz [1974] have shown that the extreme expected value of the possible payoff values can be found by using the partial average approach. Kmietowicz and Pearman [1976] concluded that the extreme variances of the possible payoff values can also be found by using the partial average approach. In other words, the extreme expected value and the extreme variances of the payoff values must occur at one of corner boundary points. However, Agunwamba [1980] pointed out a special case which was ignored by Kmietowicz and Pearman. When no more than two distinct payoff values exist for n possible states of nature, it is possible that the maximum variance occurs inside the feasible region.

Kmietowicz and Pearman [1981] then incorporated the expected value and variance into a single index, the index of utility, by introducing a coefficient of risk aversion. Under weak ranking, they showed that the maximum and minimum index of utility (extreme indexes) can usually be located at one of the corner boundary points. When no more than two distinct payoff values exist for n possible states of nature, the minimum index of utility could be located inside the feasible region, and can be found by solving a set of linear equations.

Although the coefficient of risk aversion was assumed to be known by Kmietowicz and Pearman [1981], a methodology for determining the appropriate value of the coefficient of risk aversion was not provided by these authors. Because of the unknown value of the coefficient of risk aversion, the process of applying the extreme index of utility when comparing mutually exclusive alternatives is rendered ineffective.

Constant [1983] developed a decision procedure to form an optimal final decision line in the context of uncertainty (assuming equal probabilities) for one set of mutually exclusive alternatives. In the decision procedure, Constant related the coefficient of risk aversion to the product of the minimum attractive rate of return and an angular coefficient. The possible ranges of the angular coefficient for each pair of alternatives was found by using the indiscernible region of the rate of return under various possible states of nature. A common range of the angular coefficient was identified to represent the assembly of all possible values of the angular coefficient from which a decision maker can choose. The minimum angular coefficient was suggested by Constant to form the final decision line.

In Chapter V, the decision procedure developed by Constant was simplified. The resulting simplified decision procedure has been applied to both the context of uncertainty and context of risk by this author.

One of the objectives of this research is to develop a decision procedure to form the final decision line in the context of incomplete knowledge for weak ranking. Several modifications must be made for the simplified decision procedure developed in Chapter V before it can be applied to the context of incomplete knowledge.

Section A applies the proof of Kmietowicz and Pearman [1981] which states that a necessary and sufficient condition for a relative minimum index of utility to exist inside the feasible region is that there are at most two distinct payoff values for n states of nature ($n \geq 3$). Therefore, the global minimum must occur at the corner boundary points if there are more than two distinct payoff values.

If there are two distinct payoff values, multiple relative minimums of equal value can exist. The necessary conditions for the existence of at least one relative minimum inside the feasible region are developed. If the necessary conditions are not met, the global minimum must occur at one of the corner boundary points.

If the necessary conditions are met, it is possible that a relative minimum(s) exists inside the feasible region. The multiple relative minimums can be located by a pair of linear equations. Any solution to the pair of linear equations results in a relative minimum. The multiple relative minimums have a common value of the index of utility which can be calculated directly. However, only the relative minimum(s) inside the feasible region defines the global minimum(s). In case that none of the multiple relative minimums is located inside the feasible region, the global minimum must occur at one of the corner boundary points.

Section B examines the fact that a range of rates of return on index of utility (utility index ROR) exists for a fixed angular coefficient. For a fixed angular coefficient, the largest utility index ROR is the interest rate at which the maximum index of utility is equal to zero. The smallest utility index ROR for a fixed angular coefficient is the interest rate at which the minimum index of utility also equals zero. If the maximum and minimum index of utility can be identified by comparing the values of the index of utility at various corner boundary points, then the largest and smallest utility index ROR can also be identified by comparing the utility index ROR at various corner boundary points.

In Section C, the method to identify the possible ranges of the angular coefficient for each pair of alternatives is studied. The minimum angular coefficient, $\underline{a}_{min}$, is the angular coefficient at which the smallest utility index ROR is equal to the minimum rate of return on the line of indiscernibility. After the possible ranges for all pairs of alternatives are found, the common range of the angular coefficient can be identified to represent the assembly of all possible values of the angular coefficient from which a decision maker can choose. The minimum angular coefficient of the common range (most conservative) was suggested to form the final decision line. If the pair of alternatives on which the final angular coefficient is based is not included on the final decision line, adjustment of the selected final angular coefficient is necessary.

By including the necessary modifications, Section D summarizes the decision procedure under weak ranking in the context of incomplete knowledge with a numerical example.

A. Extreme Indexes of Utility under Weak Ranking

Under the constraints of weak ranking, numerous sets of probability combinations can be formed satisfying the preordering requirements of probability. Each set of probability combinations yields a unique expected mean and variance of payoff values from possible states of nature. With a fixed value of coefficient of risk aversion, each set of probability combinations also yields a unique index of utility. Kmietowicz and Pearman [1981] have shown that the extreme values (maximum and minimum) of the index of utility for a fixed value of coefficient of risk aversion can be found at one of the corner boundary points of the feasible region if there are more than two distinct payoff values for n ($n \geq 3$) states of nature. The global minimum index of utility may occur inside the feasible region if there are one or two payoff values for n states of nature. If there is only one payoff value, there is always only one value of the index of utility which equals the payoff value since the variance is always zero.

Even if there are only two distinct payoff values for n ($n \geq 3$) states of nature, the global maximum index index of utility can still be identified by comparing the values of the index of utility at the corner boundary points. However, the relative minimum index can be located by solving the following two linear equations [Kmietowicz and Pearman 1981].

$$\sum_{i=1}^{n} Y_i \, Q_i = \frac{1 + b\,(X_t + X_1)}{2b} \qquad \text{(Eq. 6-1)}$$

$$\sum_{i=1}^{n} i \, Q_i = 1 \qquad \text{(Eq. 6-2)}$$

$$Q_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \qquad \text{(Eq. 6-3)}$$

where: n = number of possible states of nature

$Y_i = \sum_{j=1}^{i} X_j$
= sum of payoff values up to state of nature i

$Q_i = P_i - P_{i+1}$
= difference in probabilities for two consecutive states of nature

b = coefficient of risk aversion

X_1 = payoff value of state of nature 1

X_t = the second distinct payoff value which first occurs for state of nature t
X_j = payoff value of state of nature j, $X_j = X_1$ or $X_j = X_t$
P_i = probability of state of nature i

By subtracting X_1 times Eq. 6-2 from Eq. 6-1, Eq. 6-1 becomes,

$$\sum_{i=1}^{n} Y_i Q_i - X_1 \left(\sum_{i=1}^{n} i Q_i \right) = [b(X_t + X_1) + 1]/2b - X_1$$

Let a_i be defined as the number of occurrences that $X_j = X_1$ for j = 1, 2, .., i. Then Y_i could be rewritten as $Y_i = a_i X_1 + (i - a_i) X_t$.

$$\sum_{i=1}^{n} [a_i X_1 + (i - a_i) X_t] Q_i - \sum_{i=1}^{n} i X_1 Q_i = [b(X_t + X_1) + 1]/2b - X_1$$

$$\sum_{i=1}^{n} \{[a_i X_1 + (i - a_i) X_t] - i X_1\} Q_i = [bX_t + bX_1 + 1 - 2bX_1]/2b$$

$$\sum_{i=1}^{n} \{(i - a_i) X_t - (i - a_i) X_1\} Q_i = [b(X_t - X_1) + 1]/2b$$

$$\sum_{i=1}^{n} (i - a_i)(X_t - X_1) Q_i = [b(X_t - X_1) + 1]/2b$$

$$\sum_{i=1}^{n} (i - a_i)(X_t - X_1) Q_i = (X_t - X_1)/2 + 1/2b$$

$$\sum_{i=1}^{n} (i - a_i) Q_i = 1/2 + 1/2b(X_t - X_1) \qquad \text{(since } X_1 \neq X_1\text{)}$$

$$\sum_{i=1}^{t-1} (i - a_i) Q_i + \sum_{i=t}^{n} (i - a_i) Q_i = 1/2 + 1/2b(X_t - X_1)$$

$$\sum_{i=t}^{n} (i - a_i) Q_i = 1/2 + 1/2b(X_t - X_1) \qquad \text{(since } a_i = i \text{ for } i = 1, .., t-1\text{)}$$

Eq. 6-1 is replaced by Eq. 6-4.

$$\sum_{i=t}^{n} (i-a_i)Q_i = 1/2 + 1/2b(X_t-X_1) \qquad \text{(Eq. 6-4)}$$

By subtracting Eq. 6-4 from Eq. 6-2, Eq. 6-2 becomes,

$$\sum_{i=1}^{n} iQ_i - \sum_{i=t}^{n} (i-a_i)Q_i = 1 - 1/2 - 1/2b(X_t-X_1)$$

$$\sum_{i=1}^{t-1} iQ_i + \sum_{i=t}^{n} iQ_i - \sum_{i=t}^{n} iQ_i + \sum_{i=t}^{n} a_iQ_i = 1/2 - 1/2b(X_t-X_1)$$

$$\sum_{i=1}^{t-1} iQ_i + \sum_{i=t}^{n} a_iQ_i = 1/2 - 1/2b(X_t-X_1)$$

Since $a_i = i$ for $i = 1, 2, \ldots, t-1$, Eq. 6-2 is replaced by Eq. 6-5.

$$\sum_{i=1}^{n} a_iQ_i = 1/2 - 1/2b(X_t-X_1) \qquad \text{(Eq. 6-5)}$$

Eq. 6-3 remains as follows,

$$Q_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \qquad \text{(Eq. 6-3)}$$

If there are two distinct payoff values for n states of nature, any solution to Eq. 6-4 and Eq. 6-5 results in a relative minimum index of utility. Since there are multiple solutions for Eqs. 6-4 and 6-5, multiple relative minimum indexes of utility can be obtained from Eqs. 6-4 and 6-5. However, with the constraints of Eq. 6-3, the multiple relative minimum indexes of utility may not be located inside the feasible region.

It is important to be able to identify the conditions for which at least one relative minimum index of utility exists inside the feasible region. Since $Q_i \geq 0$, $a_i > 0$, and $(i-a_i) \geq 0$ for $i = 1, 2, \ldots, n$, the left hand side of both Eqs. 6-4 and 6-5 is greater than or equal to zero. The corresponding right hand side of these two equations must also be greater than or equal to zero. Therefore, $1/2 + 1/2b(X_t - X_1) \geq 0$ and $1/2 - 1/2b(X_t - X_1) \geq 0$ are necessary to have at least one relative minimum index of utility inside the feasible region. The necessary conditions can be rewritten as

$$- 1/2 \leq 1/2b(X_t - X_1) \leq 1/2 \qquad \text{(Eq. 6-6)}$$

Despite the fact that the necessary conditions, Eq. 6-6, are met, it is still possible that none of the multiple relative minimum index of utility exists inside the feasible region. In other words, the necessary conditions indicate that there is no relative minimum index of utility inside the feasible region if they are not met. But the fulfillment of the necessary conditions does not guarantee at least one relative minimum index of utility inside the feasible region.

A basic solution to Eqs. 6-4 and 6-5 has at most two unknowns which are not equal to zero. There are (n)(n-1)/2 basic solutions to Eqs. 6-4 and 6-5. Any linear combinations of these basic solutions is also a solution to Eqs. 6-4 and 6-5. Therefore, unless all the basic solutions are outside the feasible region, at least one relative minimum index of utility occurs inside the feasible region.

Although multiple relative minimum indexes of utility can be located by solving Eq. 6-4 and Eq. 6-5, they result in a common value. Therefore, any solution from Eqs. 6-4 and 6-5, is sufficient in finding the common value of the multiple relative minimum indexes of utility.

Since there are two equations and n ($n \geq 3$) unknowns, a basic solution to Eqs. 6-4 and 6-5 has at most two unknowns which are not equal to zero. It is then reasonable to assume that only Q_1 and Q_t are not equal to zero while setting all other unknowns equal to zero. With the above assumption, Eqs. 6-4, 6-5, and 6-3 can be rewritten as follows,

$$(t-a_t)Q_t = 1/2 + 1/2b(X_t-X_1)$$

$$a_1Q_1 + a_tQ_t = 1/2 - 1/2b(X_t-X_1)$$

$$Q_1 \geq 0, \quad Q_t \geq 0$$

Since the payoff values for the states of nature from one to t-1 are all equal to X_1 according to the definition of t, then $a_t = t-1$. Therefore,

$$Q_t = 1/2 + 1/2b(X_t-X_1) \qquad \text{(Eq. 6-7)}$$

$$\begin{aligned} Q_1 &= 1/2 - 1/2b(X_t-X_1) - (t-1)Q_t \\ &= 1/2 - 1/2b(X_t-X_1) - (t-1)/2 - (t-1)/2b(X_t-X_1) \\ &= 1 - t/2 - t/2b(X_t-X_1) \qquad \text{(Eq. 6-8)} \end{aligned}$$

The expressions for Q_1 and Q_t in Eq. 6-7 and Eq. 6-8 represent one of the basic solutions. This solution is feasible if both Q_1 and Q_t are positive. The solution is infeasible, (in other words, it is outside the feasible region) if either Q_1 or Q_t is negative. Whether or not the resulting values of Q_1 and Q_t are feasible or infeasible, the common value of the minimum index of utility can be evaluated from these values of Q_1 and Q_t. The probabilities for all the states of nature can be calculated once the values of Q_1 and Q_t are known.

$$P_n = Q_n = 0$$

$$P_{n-1} = P_n + Q_{n-1} = 0$$

$$\vdots$$

$$P_t = P_{t+1} + Q_t = 0 + \frac{1}{2} + \frac{1}{2b\,(X_t - X_1)}$$

$$P_{t-1} = P_t + Q_{t-1} = \frac{1}{2} + \frac{1}{2b\,(X_t - X_1)} + 0$$

$$\vdots$$

$$P_2 = P_3 + Q_2 = \frac{1}{2} + \frac{1}{2b\,(X_t - X_1)} + 0$$

$$P_1 = P_2 + Q_1 = \frac{1}{2} + \frac{1}{2b\,(X_t - X_1)} + 1 - \frac{t}{2} - \frac{t}{2b\,(X_t - X_1)}$$

$$= 1 - \frac{t-1}{2} - \frac{t-1}{2b\,(X_t - X_1)}$$

Let $\underline{P}$ be the sum of the probabilities associated with all the states of nature having X_1 as payoff. Then,

$$\underline{P} = 1 - P_t \qquad \text{(since } X_i = X_1 \text{ for } i = 1, 2, \ldots, t-1)$$

$$= 1 - \frac{1}{2} - \frac{1}{2b\,(X_t - X_1)}$$

$$= \frac{1}{2} - \frac{1}{2b\,(X_t - X_1)}$$

The index of utility can be rewritten in terms of $\underline{P}$.

$$I = EXP + b * VAR$$

$$= \sum_{i=1}^{n} P_i * X_i + b \left\{ \sum_{i=1}^{n} P_i * X_i^2 - \left[\sum_{i=1}^{n} P_i * X_i \right]^2 \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b \left\{ \underline{P}X_1^2 + (1-\underline{P})X_t^2 - \left[\underline{P}X_1 + (1-\underline{P})X_t \right]^2 \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b \left\{ \underline{P}X_1^2 + (1-\underline{P})X_t^2 - \underline{P}^2X_1^2 - 2\underline{P}(1-\underline{P})X_1X_t - (1-\underline{P})^2X_t^2 \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b \left\{ \underline{P}X_1^2(1-\underline{P}) + (1-\underline{P})X_t^2[1-(1-\underline{P})] - 2\underline{P}(1-\underline{P})X_1X_t \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b \left\{ \underline{P}(1-\underline{P})X_1^2 + \underline{P}(1-\underline{P})X_t^2 - 2\underline{P}(1-\underline{P})X_1X_t \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b\underline{P}(1-\underline{P}) \left\{ X_1^2 + X_t^2 - 2X_1X_t \right\}$$

$$= \underline{P}X_1 + (1-\underline{P})X_t + b\underline{P}(1-\underline{P})(X_t - X_1)^2$$

Substitute the expression for $\underline{P}$ into the equation,

$$I = \left[\frac{1}{2} - \frac{1}{2b(X_t-X_1)}\right]X_1 + \left[\frac{1}{2} + \frac{1}{2b(X_t-X_1)}\right]X_t + b\left[\frac{1}{4} - \frac{1}{4b^2(X_t-X_1)^2}\right](X_t-X_1)^2$$

$$= \frac{X_1+X_t}{2} + \frac{X_t-X_1}{2b(X_t-X_1)} + \frac{b(X_t-X_1)^2}{4} - \frac{b}{4b^2}$$

$$= \frac{X_1+X_t}{2} + \frac{1}{2b} + \frac{b(X_t-X_1)^2}{4} - \frac{1}{4b}$$

$$= \frac{2b(X_1+X_t) + b^2(X_t-X_1)^2 + 2 - 1}{4b}$$

$$= \frac{b^2(X_t-X_1)^2 + 2b(X_t-X_1) + 1 + 4bX_1}{4b}$$

$$= \frac{[b(X_t-X_1)+1]^2}{4b} + X_1 \qquad \text{(Eq. 6-9)}$$

If there are only two distinct payoff values, Eq. 6-9 is the common value of the multiple relative minimum indexes of utility. Notice that this common value becomes the global minimum index of utility only if at least one relative minimum exists inside the feasible region.

Thus it may be concluded that the assessment of the extreme indexes of utility under weak ranking must proceed in the following way.

a. Global maximum At least one global maximum occurs at the corner boundary points despite the number of distinct payoff values.

b. Global minimum

   i. If there are more than two distinct payoff values, the global minimum index of utility must occur at the corner boundary points.

ii. If there are only two distinct payoff values, it is possible that relative minimums exist inside the feasible region. Check the necessary conditions using Eq. 6-6. If the necessary conditions are not met, the global minimum will still occur at the corner boundary points. If the necessary conditions are met, Eqs. 6-4, 6-5, and 6-3 are served to locate the relative minimums. Check all basic solutions to Eqs. 6-4 and 6-5.

Case 1: If at least one basic solution of Eqs. 6-4 and 6-5 is feasible, the global minimum index of utility can be calculated by Eq. 6-9.

Case 2: If all the basic solutions to Eqs. 6-4 and 6-5 are not feasible, the global minimum index of utility must occur at the corner boundary points.

The same numerical example used in Chapter V will be adopted again to demonstrate the application of the decision procedure in the context of incomplete knowledge.

For alternative A compared to present conditions, the four payoff values (AEX) for the various states of nature using a rate of return of 20% were as follows:

N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = 156.20$
N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 277.08$
N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = 145.04$
N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 388.14$

Since all four payoff values are different from one another, the extreme values of the index of utility can be found at the corner boundary points. Assuming the angular coefficient is known to be -0.05 (or equivalently the coefficient of risk aversion is -0.01), the values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	1.0	-	-	-	156.20	0	156.20
2	.50	.50	-	-	216.64	3652.79	180.11**
3	.33	.33	.33	-	192.78	3574.36	157.03
4	.25	.25	.25	.25	241.62	9837.09	143.25*

Since the coefficient of risk aversion is defined as the product of the angular coefficient and the minimum attractive rate of return, different values for the coefficient of risk aversion result at different interest rates by keeping the angular coefficient at the same value of -0.05. For each value of the coefficient of risk aversion, both the maximum(**) and minimum(*) index of utility can be identified by the procedure described above. Figure 2 shows the extreme values of the index of utility for a spectrum of interest rates with an angular coefficient of -0.05. Since there are more than two distinct payoff values at each of the interest rates, the maximum and minimum index of utility always occur at the corner boundary points for the numerical example.

B. Maximum and Minimum Utility Index Rates of Return

In the context of uncertainty, Constant [1983] defined the utility index ROR at an angular coefficient as the rate of return that causes the index of utility to equal zero. Since there is only one value of index of utility in respect to one value of rate of return (or one value of coefficient of risk aversion) for a fixed angular coefficient in the

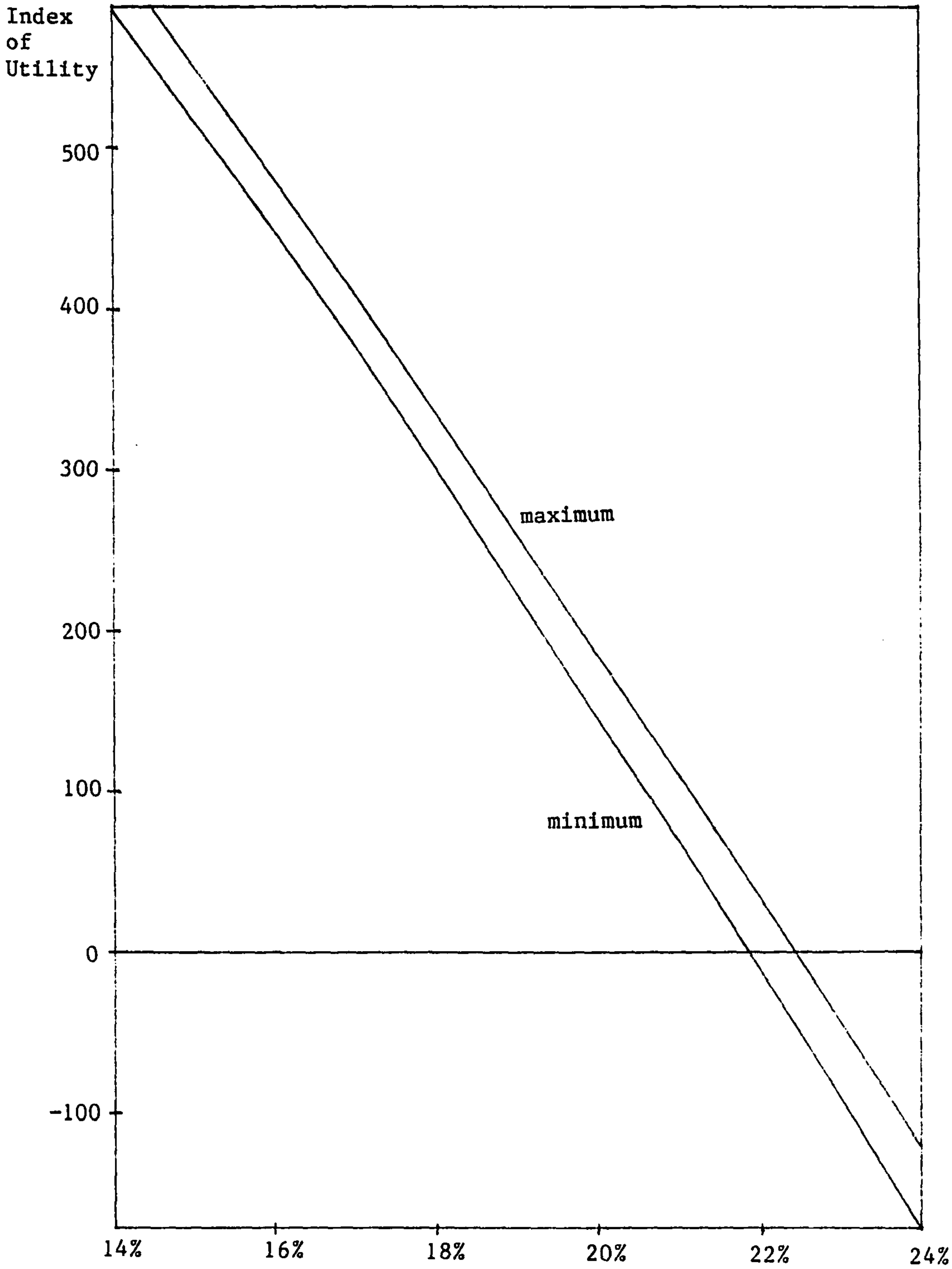


Figure 2. Extreme values of the index of utility under weak ranking

context of uncertainty, there is only one rate of return that will cause the index of utility to equal zero. In other words, only one utility index ROR exists in the context of uncertainty for each value of angular coefficient.

For weak ranking in the context of incomplete knowledge, there is a range of values for the index of utility in respect to one value of rate of return at a fixed angular coefficient as discussed in Section A. The approach to search for the maximum and minimum index of utility in this range was presented in Section A. Since there is a range of values for the index of utility in respect to one value of rate of return at a fixed angular coefficient, there must be multiple rates of return at which the index of utility of zero value is included in the possible range of index values. In other words, there must be a range of utility index ROR for a fixed angular coefficient which may cause the index of utility to equal zero at certain probability combinations.

Let the upper limit of the range of utility index ROR be defined as the maximum utility index ROR, and the lower limit as the minimum utility index ROR. For a fixed angular coefficient, the maximum utility index ROR is the interest rate at which the maximum index of utility equals zero. The minimum utility index ROR for a fixed angular coefficient is the interest rate at which the minimum index of utility equals zero.

At least one global maximum index of utility occurs at the corner boundary points despite the number of distinct payoff values as discussed in Section A. Therefore, the maximum utility index ROR must occur at one of the corner boundary points. The value of the maximum utility index ROR can be found by the trial and error method. At each trial and error routine, the values of index of utility for the corner boundary points are calculated at the trial rate of return. The trial and error routine is repeated until a utility index ROR is found at which the maximum index of utility is of zero value.

However, it is possible that the global minimum utility index ROR occurs inside the feasible region. A trial and error routine is still used to search for the value of the minimum utility index ROR. In each trial and error routine, the payoff values for the various states of nature are calculated according to the trial rate of return.

If there are more than two distinct payoff values, the minimum index of utility must occur at the corner boundary points. In this case, it is only necessary to calculate the values of index of utility at the corner boundary points in order to determine the global minimum index of utility.

If there are only two distinct payoff values, it is possible that a relative minimum index of utility exists inside the feasible region. The procedure is to check the necessary conditions: $-1/2 \leq 1/2b(X_t-X_1) \leq 1/2$. If the necessary conditions are not met, the global minimum index of utility must occur at the corner boundary points.

If the necessary conditions are met, Eqs. 6-4, 6-5, and 6-3 are served to locate the relative minimums. Check all basic solutions to Eqs. 6-4 and 6-5. If at least one basic solution of Eqs. 6-4 and 6-5 is feasible, the global minimum index of utility can be calculated by Eq.

6-9. If all the basic solutions to Eqs. 6-4 and 6-5 are not feasible, the global minimum index of utility must occur at the corner boundary points.

The trial and error routine is repeated until a utility index ROR is found at which the minimum index of utility is of zero value.

In the numerical example for alternative A compared to present conditions, the maximum and minimum utility index ROR at an angular coefficient of -0.05 are found by trial and error routine to be 22.37% and 21.83%, respectively.

At 22.37%, the payoff values are as follows,

$$
\begin{aligned}
AEX_1 &= -19.88\\
AEX_2 &= 99.26\\
AEX_3 &= -30.91\\
AEX_4 &= 208.67
\end{aligned}
$$

The values of the mean, variance and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	1.0	-	-	-	-19.88	0	-19.88
2	.50	.50	-	-	39.69	3548.48	0.00**
3	.33	.33	.33	-	16.16	3473.30	-22.69
4	.25	.25	.25	.25	64.29	9554.01	-42.58

Notice that the maximum index of utility(**) which occurs at corner boundary point 2 is of zero value at the trial rate of return of 22.37% for an angular coefficient of -0.05. At 21.83%, the payoff values are as follows,

$$
\begin{aligned}
AEX_1 &= 20.53\\
AEX_2 &= 140.06\\
AEX_3 &= 9.47\\
AEX_4 &= 249.84
\end{aligned}
$$

Since there are more than two payoff values, the minimum index of utility must occur at the corner boundary points. The values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	1.0	-	-	-	20.53	0	20.53
2	.50	.50	-	-	80.30	3571.83	41.31
3	.33	.33	.33	-	56.69	3495.93	18.53
4	.25	.25	.25	.25	104.98	9617.41	0.00*

Notice that the minimum index of utility(*) which occurs at corner boundary point 4 is of zero value at the trial rate of return of 21.83% for an angular coefficient of -0.05.

C. Determination of Final Angular Coefficient

In the context of incomplete knowledge, there is a range of utility index ROR for each value of the angular coefficient. The approach to identify the maximum and minimum utility index ROR was presented in Section B. This section will focus on the selection of the angular coefficient which will form the final decision line.

Before determining the value of the final angular coefficient, the possible angular coefficients should be evaluated for each pair of alternatives. For each pair of alternatives, i.e., alternatives compared with present conditions or compared with one another, a line of indiscernibility is drawn between the highest rate of return and the lowest rate of return representing the "best" and "worst" states of nature. Since a rate of return higher than the highest or lower than the lowest is impossible, a reasonable utility index ROR must fall within the line of indiscernibility. Therefore, those angular coefficients, which could cause the utility index ROR to be greater than the highest rate of return from the "best" state of nature or less than the lowest rate of return from the "worst" state of nature, need not to be considered any further. Those angular coefficients, which could cause the utility index ROR to fall within the line of indiscernibility, are the candidates for the final angular coefficient.

The minimum angular coefficient, $\underline{a}_{min}$, is the lower limit of the possible angular coefficients at which the utility index ROR are greater than or equal to the smallest rate of return on the line of indiscernibility. It is not necessary to consider an angular coefficient less than $\underline{a}_{min}$ because any angular coefficient less than $\underline{a}_{min}$ could cause a utility index ROR to be lower than the lowest rate of return possible.

The maximum angular coefficient, $\underline{a}_{max}$, is the upper limit of the possible angular coefficients at which the utility index ROR are less than or equal to the largest rate of return on the line of indiscernibility. Theoretically, those angular coefficients between $\underline{a}_{min}$ and $\underline{a}_{max}$ are all possible for final selection. However, since the value of $\underline{a}_{max}$ is always positive, causing contradictory conclusions as discussed in Chapter V, the possible range for the angular coefficient is from $\underline{a}_{min}$ to zero.

Although the value of the minimum angular coefficient for a certain pair of alternatives can be found by the trial and error method as in Constant's decision approach, a more straightforward approach to locate the value of the minimum angular coefficient is developed. At the minimum angular coefficient, $\underline{a}_{min}$, the minimum utility index ROR is equal to the smallest rate of return on the line of indiscernibility, i_{min}. And the minimum utility index ROR for a fixed angular coefficient is the interest rate at which the minimum index of utility equals zero. With the minimum utility index ROR known to be equal to i_{min}, the angular coefficient which causes the minimum index of utility to equal zero can be found reversely.

The payoff values for all possible states of nature are first calculated according to i_{min}. If there are more than two distinct payoff values, the minimum index of utility must occur at one of the corner boundary points. The minimum angular coefficient must also occur at one of the corner boundary points. The possible minimum angular coefficient for each corner boundary point can be obtained by the following equation,

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 6-10)}$$

where: j = a certain pair of alternatives
$\underline{a}_{j,min}$ = lower limit of angular coefficient
$\underline{i}_{j,min}$ = lowest rate of return on the line of indiscernibility
$EXP[AEX]_j$ = expected value of payoffs based on $i_{j,min}$
$VAR[AEX]_j$ = variance of payoffs based on $i_{j,min}$

The procedure is to choose the largest $\underline{a}_{min}$ value from the $\underline{a}_{min}$ values of all corner boundary points as the minimum angular coefficient for this pair of alternatives. The largest $\underline{a}_{min}$ is selected because it is the smallest angular coefficient common to all corner boundary points.

If there are only two distinct payoff values, it is possible that a relative minimum index of utility exists inside the feasible region. Since the common value of the multiple minimum indexes of utility can be calculated by Eq. 6-9, the value of the coefficient of risk aversion can be obtained by setting Eq. 6-9 equal to zero.

$$I = \frac{[b(X_t-X_1)+1]^2}{4b} + X_1 = 0$$

$$\frac{b^2(X_t-X_1)^2 + 2b(X_t-X_1) + 1 + 4bX_1}{4b} = 0$$

$$b^2(X_t-X_1)^2 + 2b(X_t+X_1) + 1 = 0$$

$$(X_t-X_1)^2\, b^2 + 2(X_t+X_1)\, b + 1 = 0$$

With the payoff values (X_1 and X_t) known, the coefficient of risk aversion can be calculated according to the following equation,

$$b = \frac{-2(X_t+X_1) \pm \sqrt{[2(X_t+X_1)]^2 - 4*(X_t-X_1)^2*1}}{2*(X_t-X_1)^2}$$

$$= \frac{-2(X_t+X_1) \pm \sqrt{4X_t^2 + 4X_1^2 + 8X_tX_1 - 4X_t^2 - 4X_1^2 + 8X_tX_1}}{2(X_t-X_1)^2}$$

$$= \frac{-2(X_t+X_1) \pm \sqrt{16X_tX_1}}{2(X_t-X_1)^2}$$

$$= \frac{-2(X_t+X_1) \pm 4\sqrt{X_tX_1}}{2(X_t-X_1)^2}$$

$$= \frac{-(X_t+X_1) \pm 2\sqrt{X_tX_1}}{(X_t-X_1)^2} \qquad \text{(Eq. 6-11)}$$

Notice that there are two values of the coefficient of risk aversion obtained from Eq. 6-11. However, only the coefficient of risk aversion that satisfies all of the following tests should be used to determine the minimum angular coefficient for this pair of alternatives.

Test 1: The value of the coefficient of risk aversion must be negative.

Test 2: The necessary conditions: $-1/2 \leq 1/2b(X_t-X_1) \leq 1/2$ must be satisfied.

Test 3: At least one basic solution to Eqs. 6-4 and 6-5 must be feasible.

If both of the values of the coefficient of risk aversion obtained from Eq. 6-11 fail to satisfy either test, the global minimum index of utility must occur at one of the corner boundary points. The minimum angular coefficient must also occur at the corner boundary points.

After the possible ranges for all pairs of alternatives have been found, the common range of the angular coefficients can be identified. The lower limit of the common range of the angular coefficients is the maximum of the minimum angular coefficients, $\underline{a}_{min}$, for all pairs of alternatives. The upper limit of the common range of the angular coefficients is zero. The common range of the angular coefficients represents the assembly of all possible values of angular coefficients from which a decision maker can choose. The lower limit of the common range of the angular coefficients forms the final decision line. If the pair of alternatives on which the final angular coefficient is established is not included on the final decision line, adjustment of the selected final angular coefficient is necessary.

D. Summary

For weak ranking in the context of incomplete knowledge, the necessary steps in the simplified approach are restated as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

2. For all pairs of alternatives, i.e., alternative compared with present conditions or compared with one another, draw a line of indiscernibility between the lowest rate of return and the highest rate of return representing the "best case" and "worst case" scenarios. Let the lowest rate of return ("worst case"

scenario) be defined as i_{min}, and the highest rate of return ("best case" scenario) as i_{max}.

3. Obtain the minimum angular coefficients based on the value of i_{min} on the lines of indiscernibility according to either one of the following cases:

   a) If there are more than two distinct payoff values, calculate $\underline{a}_{min}$ for all corner boundary points according to Eq. 6-10.

$$\underline{a}_{j,min} = \frac{-\ \mathrm{EXP}[\mathrm{AEX}]_j}{(i_{j,min})(\mathrm{VAR}[\mathrm{AEX}]_j)} \qquad \text{(Eq. 6-10)}$$

   where: j = a certain pair of alternatives
   $\underline{a}_{j,min}$ = lower limit of angular coefficient
   $i_{j,min}$ = lowest rate of return on the line of indiscernibility
   $\mathrm{EXP}[\mathrm{AEX}]_j$ = expected value of payoffs based on $i_{j,min}$
   $\mathrm{VAR}[\mathrm{AEX}]_j$ = variance of payoffs merit based on $i_{j,min}$

   Choose the largest $\underline{a}_{min}$ value from among the $\underline{a}_{min}$ values of all corner boundary points as the minimum angular coefficient for each pair of alternatives.

   b) If there are only two distinct payoff values, calculate the possible values of b, the coefficient of risk aversion.

$$b = \frac{-(X_t+X_1) \pm 2\sqrt{X_t X_1}}{(X_t-X_1)^2} \qquad \text{(Eq. 6-11)}$$

   Only the value of the coefficient of risk aversion that satisfies all of the following tests should be used to determine the minimum angular coefficient for this pair of alternatives.

   Test 1: The value of the coefficient of risk aversion must be negative.

   Test 2: The necessary conditions: $-1/2 \leq 1/2b(X_t-X_1) \leq 1/2$ must be satisfied.

   Test 3: At least one basic solution to Eqs. 6-4 and 6-5 must be feasible.

   If both of the values of the coefficient of risk aversion obtained from Eq. 6-11 fail to satisfy all tests, the minimum angular coefficient must occur at the corner boundary points.

4. For all pairs of alternatives, choose the largest (in other words, the least negative) value of $\underline{a}_{j,min}$ calculated in Step 3. The range for possible final $\underline{a}$ values is then defined by the maximum of $\underline{a}_{min}$ (which is a negative value) and zero. The $\max(\underline{a}_{min})$ value is the most conservative angular coefficient within the possible range; as the angular coefficients within the range becomes less negative, decisions become less conservative.

5. Calculate the minimum utility index ROR for each pair of alternatives based on the angular coefficient selected using a trial and error routine. In each trial and error routine, the payoff values for the various states of nature are calculated.

   a) If there are more than two distinct payoff values, it is only necessary to calculate the values of index of utility at the corner boundary points in order to determine the minimum index of utility.

   b) If there are only two distinct payoff values, it is possible that solutions exist inside the feasible region. Check the necessary conditions: $-1/2 \leq 1/2b(X_t - X_1) \leq 1/2$. If the necessary conditions are not met, the global minimum will still occur at corner boundary points. If the necessary conditions are met, Eqs. 6-4, 6-5, and 6-3 are served to locate the relative minimums. Check all basic solutions to Eqs. 6-4 and 6-5. If at least one basic solution to Eqs. 6-4 and 6-5 is feasible, the global minimum index of utility can be calculated by Eq. 6-9. If all the basic solutions to Eqs. 6-4 and 6-5 are not feasible, the global minimum index of utility must occur at the corner boundary points.

   The trial and error routine is repeated until a utility index ROR is found at which the minimum index of utility is of zero value.

6. Form the final decision line by taking the minimum utility index ROR for each pair of alternatives calculated in Step 5.

   Examine the final decision line. If the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is included on the final decision line, the decision procedure is completed. However, if the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is not included on the final decision line, the final $\underline{a}_{min}$ value must be modified. Take the next least negative value of $\underline{a}_{min}$ as the revised final angular coefficient. Steps 5 and 6 are repeated as many times as necessary.

The same numerical example used for Constant's approach will be presented again to illustrate the formation of the final decision line under weak ranking. The necessary steps are as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

   For state of nature N_1, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	22.11%			
B	21.15%	18.72%		
C	20.42%	17.57%	15.24%	
D	19.06%	15.94%	14.05%	13.46%

   For state of nature N_2, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	23.68%			
B	23.57%	23.30%		
C	21.63%	18.12%	5.91%	
D	21.08%	18.45%	15.03%	18.93%

   For state of nature N_3, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	21.96%			
B	23.57%	27.33%		
C	19.22%	14.43%	-48.22%	
D	20.38%	18.81%	12.40%	24.61%

   For state of nature N_4, the rates of return are as follows:

	Internal ROR over P	A	B	C
Alternative A	25.08%			
B	22.36%	14.99%		
C	24.04%	22.29%	34.26%	
D	20.99%	16.71%	17.79%	7.11%

2. For all pairs of alternatives, i.e., alternative compared with

present conditions or compared with one another, draw a line of indiscernibility between the highest rate of return and the lowest rate of return representing the "best case" and "worst case" scenarios, respectively. Let the highest rate of return be defined as i_{max}, and the lowest rate of return as i_{min}. The lines of indiscernibility for all pairs of alternatives on the decision line are as follows:

```
A - P +--------------------------------------------******------------
                                                21.96%  25.08%

B - P +-------------------------------------------*****--------------
                                              21.15% 23.57%

B - A +------------------------------************************--------
                                  14.99%                  27.33%

C - P +---------------------------------------**********-------------
                                           19.22%   24.04%

C - A +-----------------------------*****************----------------
                                 14.43%           22.29%

C - B ***************************************************************
       0                                                        34.26%

D - P +---------------------------------------*****------------------
                                         19.06%  21.08%

D - A +---------------------------------******-----------------------
                                   15.94%   18.81%

D - B +-------------------------************-------------------------
                             12.40%      17.99%

D - C +-----*********************************************------------
          7.11%                                       24.61%
```

3. Obtain the minimum angular coefficients based on the value of i_{min} on the lines of indiscernibility.

For alternative A compared with present conditions, the value of i_{min} on the line of indiscernibility is found to be 21.96%. The following figures of merit (AEX in this case) can be calculated based on 21.96%:

N_1: e_{F1} = 0%, e_{F2} = 0%, AEX_1 = 11.05
N_2: e_{F1} = 2%, e_{F2} = 1%, AEX_2 = 130.49
N_3: e_{F1} = 2%, e_{F2} =-1%, AEX_3 = 0.00
N_4: e_{F1} = 1%, e_{F2} = 3%, AEX_4 = 240.19

Since there are more than two distinct payoff values, calculate $\underline{a}_{min}$ for all the corner boundary points according to Eq. 6-10,

$$\underline{a}_{j,min} = \frac{- EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 6-10)}$$

The resulting values of the $\underline{a}_{min}$ for each corner boundary point are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	$\underline{a}_{min}$ value
1	1.0	-	-	-	11.05	0	$-\infty$
2	.50	.50	-	-	70.77	3566.31	-0.09038
3	.33	.33	.33	-	47.18	3490.57	-0.06156
4	.25	.25	.25	.25	95.43	9602.47	-0.04526

Choose the largest $\underline{a}_{min}$ value from among the $\underline{a}_{min}$ values of all corner boundary points, -0.04526, as the minimum angular coefficient for alternative A compared with present conditions.

In the same manner, the minimum angular coefficients for all pairs of alternatives at various corner boundary points (CBP) can be calculated and listed as follows,

Pair of Alternatives	$\underline{a}_{min}$ CBP 1	CBP 2	CBP 3	CBP 4	Max.
A - P	$-\infty$	-.09038	-.06156	-.04526	-.04526
B - P	$-\infty$	-.03669	-.05504	-.06642	-.03669
B - A	$-\infty$	-.23272	-.12720	-.05831	-.05831
C - P	$-\infty$	-.21196	-.05426	-.02793	-.02793
C - A	$-\infty$	-6.9229	-.13918	-.07040	-.07040
C - B	$-\infty$	+	+	+	+
D - P	$-\infty$	-.03483	-.05576	-.07107	-.03483
D - A	$-\infty$	-.06847	-.09430	-.09225	-.06847
D - B	$-\infty$	-1.6527	-.23163	-.11484	-.11484
D - C	$-\infty$	-.50576	-.22254	-.09349	-.09349

4. From among all pairs of alternatives, choose the least negative value of $\underline{a}_{j,min}$, -0.02793 (C over P). This value is the most conservative angular coefficient within the possible range from -0.02793 to zero.

5. Calculate the minimum utility index ROR for each pair of alternatives based on the angular coefficient selected using a

trial and error routine. In each trial and error routine, the payoff values for the various states of nature are calculated according to a trial rate of return.

For alternative A compared to present conditions, the four payoff values (AEX) for the various states of nature using a trial rate of return of 22.11% were as follows:

N_1: e_{F1} = 0%, e_{F2} = 0%, AEX_1 = 0.00
N_2: e_{F1} = 2%, e_{F2} = 1%, AEX_2 = 119.33
N_3: e_{F1} = 2%, e_{F2} =-1%, AEX_3 = -11.05
N_4: e_{F1} = 1%, e_{F2} = 3%, AEX_4 = 228.92

Since all the four payoff values are different from one another, the extreme values of the index of utility can be found at the corner boundary points. Using the angular coefficient of -0.02793, the values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	1.0	-	-	-	0.00	0	0.00*
2	.50	.50	-	-	59.66	3559.94	37.68
3	.33	.33	.33	-	36.09	3484.40	14.58
4	.25	.25	.25	.25	84.30	9585.08	25.12

Since the minimum index of utility(*) is zero at the trial rate of return of 22.11%, this trial rate of return is the minimum utility index ROR for alternative A compared to present conditions at the angular coefficient of -0.02793.

In the same manner, the minimum utility index ROR for all the pairs of alternatives can be calculated and summarized as follows,

	Minimum utility index ROR over P	A	B	C
Alternative A	22.11%			
B	21.15%	17.86%		
C	19.22%	16.23%	0.20%	
D	19.06%	15.94%	13.66%	12.11%

6. The final decision line based on the angular coefficient selected, i.e., -0.02793, is as follows,

```
                    D                      B          A         P
+---------------------------------------+----------+----------+-------
0                                    13.66%     17.86%     22.11%
```

Since the final decision line does not include the pair of alternatives C over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.03483 (D over P).

5a. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.03483.

	Minimum utility index ROR over			
	P	A	B	C
Alternative A	22.11%			
B	21.15%	17.14%		
C	18.74%	16.11%	0.18%	
D	19.06%	15.94%	13.61%	11.32%

6a. The final decision line based on the angular coefficient selected, i.e., -0.03483, is as follows,

```
                 D                  B        A       P
+-------------------------------+---------+----------+-------
0                            13.61%   17.14%     22.11%
```

Since the final decision line does not include the pair of alternatives D over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.03669 (B over P).

5b. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.03669.

	Minimum utility index ROR over			
	P	A	B	C
Alternative A	22.11%			
B	21.15%	16.95%		
C	18.61%	16.06%	0.18%	
D	19.01%	15.94%	13.60%	11.12%

6b. The final decision line based on the angular coefficient selected, i.e., -0.03669, is as follows,

```
                 D                  B        A       P
+-------------------------------+---------+----------+-------
0                            13.60%   16.95%     22.11%
```

Since the final decision line does not include the pair of alternatives B over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.04526 (A over P).

5c. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.04526.

	Minimum utility index ROR over P	A	B	C
Alternative A	21.96%			
B	20.87%	16.13%		
C	18.04%	15.62%	0.16%	
D	18.77%	15.94%	13.54%	10.30%

6c. The final decision line based on the angular coefficient selected, i.e., -0.04526, is as follows,

```
                D                    B          A       P
+-------------------------------+--------+-----------+--------
0                            13.54%  16.13%       21.96%
```

Since the final decision line does include the pair of alternatives A over P, the final decision line is completed.

VII. DECISION PROCEDURE FOR STRICT RANKING IN THE CONTEXT OF INCOMPLETE KNOWLEDGE

For strict ranking in the context of incomplete knowledge, Kmietowicz and Pearman [1981] have shown that the extreme expected values must occur at corner boundary points. In Chapter III, the algorithm of searching for the maximum and minimum variances under strict ranking was demonstrated. The expected value and variance were then combined into a single index, the index of utility, by introducing a coefficient of risk aversion. In Chapter IV, the algorithm of searching for the maximum and minimum index of utility under strict ranking was studied. Under strict ranking, it was shown that the global maximum index of utility always occurs at the corner boundary points. The global minimum index of utility can usually be located at one of the corner boundary points. When no more than two distinct payoff values exist for n possible states of nature, the global minimum index of utility may occur inside the feasible region.

A decision procedure has been developed in Chapter VI to form the final decision line in the context of incomplete knowledge for weak ranking. One of the objectives of this research is to apply the decision procedure to form the final decision line in the context of incomplete knowledge for strict ranking.

Section A applies the proof in Chapter IV which states that a necessary and sufficient condition for a relative minimum index of utility to exist inside the feasible region is that there are at most two distinct payoff values for n states of nature ($n \geq 3$).

If there are two distinct payoff values, multiple relative minimum indexes of utility of equal value exist. The multiple relative minimum indexes of utility may be located either inside or outside the feasible region. If at least one relative minimum index of utility exists inside the feasible region, the relative minimum defines the global minimum index of utility. If none of the multiple relative minimums exists inside the feasible region, the global minimum must occur at one of the corner boundary points.

Section B examines the fact that a range of rates of return on index of utility (utility index ROR) exists for a fixed angular coefficient. For a fixed angular coefficient, the largest utility index ROR is the interest rate at which the maximum index of utility equals zero. The smallest utility index ROR is the interest rate at which the minimum index of utility also equals zero. If the maximum and minimum index of utility can be identified by comparing the values of the index of utility at various corner boundary points, then the largest and smallest utility index ROR can also be identified by comparing the utility index ROR at various corner boundary points.

In Section C, the method to identify the possible ranges of the angular coefficient for each pair of alternatives is studied. The minimum angular coefficient, $\underline{a}_{min}$, is the angular coefficient at which the smallest utility index ROR is equal to the minimum rate of return of the indiscernible region. After the possible ranges for all pairs of alternatives are found, the common range of the angular coefficient can be identified to represent the assembly of all possible values of the angular coefficient from which a decision maker can choose. The minimum angular coefficient is suggested to form the final decision line. If the pair of alternatives on which the final angular coefficient is based is not included on the final decision line, adjustment of the selected final angular coefficient is necessary.

By including the necessary modifications, Section D summarizes the decision procedure under strict ranking in the context of incomplete knowledge with a numerical example.

A. Extreme Values of Index of Utility

Under the constraints of strict ranking of incomplete knowledge, it was proved in Chapter IV that the extreme indexes of utility (maximum and minimum) for a coefficient of risk aversion can be found at one of the corner boundary points if there are more than two distinct payoff values for n ($n \geq 3$) states of nature. The global minimum index of utility may occur inside the boundary if there are one or two payoff values for n states of nature. If there is only one payoff value, there is only one value of the index of utility which equals the payoff value since the variance is always zero.

If there are only two distinct payoff values for n ($n \geq 3$) states of nature, the maximum index can still be identified by comparing the values of index of utility at the corner boundary points as discussed in Section C of Chapter IV. However, multiple relative minimum indexes of utility which are located inside the feasible region can be located by solving the following equations (Eqs. 4-15, 4-16, and 4-3 in Chapter IV).

$$\sum_{i=t}^{n} (1-a_i)T_i = 1/2 - C'' + 1/2b(X_t-X_1) \quad \text{(Eq. 7-1)}$$

$$\sum_{i=1}^{n} a_i T_i = 1/2 - C' - 1/2b(X_t-X_1) \quad \text{(Eq. 7-2)}$$

$$T_i \geq 0 \quad (\text{for } i = 1, 2, \ldots, n) \quad \text{(Eq. 7-3)}$$

The multiple relative minimum indexes can be located either inside or outside of the feasible region. The necessary conditions for at least one relative minimum index of utility to exist inside the feasible are:

$$- 1/2 + C'' \leq 1/2b(X_t-X_1) \leq 1/2 - C' \quad \text{(Eq. 7-4)}$$

If none of the multiple relative minimum indexes of utility exists inside the feasible region, the global minimum index of utility still occurs at one of the corner boundary points. If at least one relative minimum index of utility exists inside the feasible region, the global minimum index of utility can be calculated directly by the following equation (Eq. 4-23 in Chapter IV).

$$I = \frac{[b(X_t-X_1) + 1]^2}{4b} + X_1 \quad \text{(Eq. 7-5)}$$

In summary, the assessment of the extreme values of the index of utility under strict ranking in the context of incomplete knowledge must proceed in the following way.

a. Global maximum At least one global maximum occurs at the corner boundary points despite the number of distinct payoff values.

b. Global minimum

i. If there are more than two distinct payoff values, the global minimum index of utility must occur at the corner boundary points.

ii. If there are only two distinct payoff values, it is possible that relative minimum indexes of utility exist inside the feasible region. Check the necessary conditions using Eq. 7-4. If the necessary conditions are not met, the global minimum index of utility will still occur at the corner boundary points. If the necessary conditions are not met, Eqs. 7-1, 7-2, and 7-3 are served to locate the multiple relative minimum indexes of utility. Check all basic solutions to Eqs. 7-1 and 7-2.

Case 1: If at least one basic solution is feasible, the global minimum index of utility can be calculated by Eq. 7-5.

Case 2: If none of the basic solutions is feasible, the global minimum index of utility must occur at the corner boundary points.

The same numerical example used in Chapter V will be adopted again with additional data as required under strict ranking. The additional data are:

$k_1 = .08$, $k_2 = .06$, $k_3 = .04$, $k_4 = .02$

For alternative A compared to present conditions, the four payoff values (AEX) for the various states of nature using a rate of return of 20% were as follows,

N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = 156.20$
N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 277.08$
N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = 145.04$
N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 388.14$

Since all four payoff values are different from one another, the extreme indexes of utility can be found at the corner boundary points. Assuming the angular coefficient is known to be -0.05 (or equivalently the coefficient of risk aversion is -0.01), the values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature				Mean	Variance	Index of utility
	1	2	3	4			
1	.80	.12	.06	.02	174.68	2495.40	149.72
2	.50	.42	.06	.02	210.94	4223.87	168.70**
3	.40	.32	.26	.02	196.62	4150.24	155.12
4	.35	.27	.21	.17	225.93	8225.00	143.38*

Since the coefficient of risk aversion is defined as the product of the angular coefficient and the minimum attractive rate of return, different values for the coefficient of risk aversion result at different interest rates by keeping the angular coefficient at the same value of -0.05. For each value of the coefficient of risk aversion, both the maximum(**) and minimum(*) index of utility can be identified by the procedure described above. In this particular numerical example, both the maximum and minimum index of utility occur at the corner boundary points for different values of the coefficient of risk aversion. Figure 3 shows the extreme values of the index of utility for a spectrum of interest rates with an angular coefficient of -0.05. Since there are more than two distinct payoff values at each of the interest rates, the maximum and minimum index of utility always occur at the corner boundary points for the numerical example.

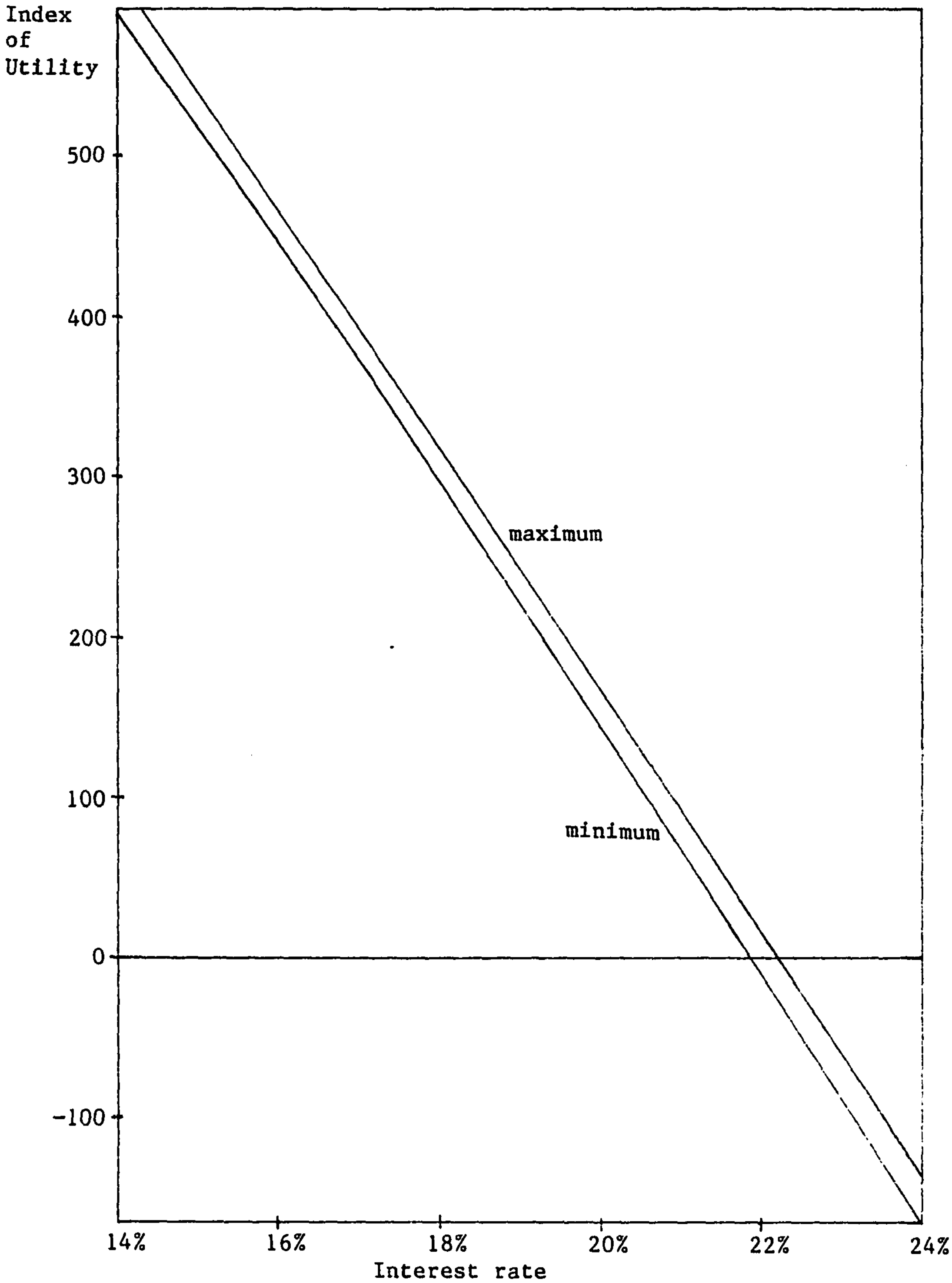


Figure 3. Extreme values of the index of utility under strict ranking

B. Maximum and Minimum Utility Index Rates of Return

For strict ranking in the context of incomplete knowledge, there is a range of values for the index of utility in respect to one value of rate of return at a fixed angular coefficient as discussed in Section A. The approach to search for the maximum and minimum index of utility in this range was also presented in Section A. Since there is a range of values for the index of utility in respect to one value of rate of return at a fixed angular coefficient, there must be multiple rates of return at which the index of utility of zero value is included in the possible range of index values. In other words, there must be a range of utility index ROR for a fixed angular coefficient which may cause the index of utility to equal zero at certain probability combinations. Let the upper limit of the range of utility index ROR be defined as the maximum utility index ROR, and the lower limit as the minimum utility index ROR. For a fixed angular coefficient, the maximum utility index ROR is the interest rate at which the maximum index of utility equals zero. The minimum utility index ROR is the interest rate at which the minimum index of utility equals zero.

At least one global maximum index of utility occurs at the corner boundary points despite the number of distinct payoff values as discussed in Section A. Therefore, the maximum utility index ROR must occur at one of the corner boundary points. The value of the maximum utility index ROR can be found by a trial and error routine. At each trial and error routine, the values of index of utility on the corner boundary points are calculated at a trial rate of return. The trial and error routine is repeated until a utility index ROR is found at which the maximum index of utility is of zero value.

However, it is possible that the minimum utility index ROR occurs inside the feasible region. A trial and error routine is still used to search for the value of the minimum utility index ROR. In each trial and error routine, the payoff values for various states of nature are calculated according to a trial rate of return.

If there are more than two distinct payoff values, the minimum index of utility must occur at the corner boundary points. In this case, it is only necessary to calculate the values of index of utility at the corner boundary points in order to determine the minimum index of utility.

If there are only two distinct payoff values, it is possible that a relative minimum index of utility exists inside the feasible region. The procedure is to check the necessary conditions: $-1/2 + C'' \leq 1/2b(X_t - X_1) \leq 1/2 - C'$ (Eq. 7-4). If the necessary conditions are not met, the global minimum index of utility must occur at corner boundary points. If the necessary conditions are met, Eqs. 7-1, 7-2, and 7-3 are served to locate the multiple relative minimum indexes of utility. Check all basic solutions to Eqs. 7-1 and 7-2. If at least one basic solution is feasible, the global minimum index of utility can be calculated by Eq. 7-5. If none of the basic solutions is feasible, the global minimum index of utility must occur at corner boundary points.

The trial and error routine is repeated until a utility index ROR is found at which the minimum index of utility is of zero value.

In the numerical example for alternative A compared to present conditions, the maximum and minimum utility index ROR at an angular coefficient of -0.05 are found by trial and error routine to be 22.22% and 21.85%, respectively.

At 22.22%, the payoff values are as follows,

$$AEX_1 = -8.33$$
$$AEX_2 = 110.92$$
$$AEX_3 = -19.37$$
$$AEX_4 = 220.44$$

The values of the mean, variance and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	.80	.12	.06	.02	9.89	2428.37	-17.08
2	.50	.42	.06	.02	45.67	4110.83	0.00**
3	.40	.32	.26	.02	31.54	4039.70	-13.34
4	.35	.27	.21	.17	60.44	8032.67	-28.79

Notice that the maximum index of utility(**) which occurs at corner boundary point 2 is of zero value at the trial rate of return of 22.22% for an angular coefficient of -0.05. At 21.85%, the payoff values are as follows,

$$AEX_1 = 19.22$$
$$AEX_2 = 138.74$$
$$AEX_3 = 8.16$$
$$AEX_4 = 248.51$$

Since there are more than two payoff values, the minimum index of utility must occur at the corner boundary points. The values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	.80	.12	.06	.02	37.49	2439.34	10.84
2	.50	.42	.06	.02	73.35	4129.32	28.23
3	.40	.32	.26	.02	59.18	4057.78	14.85
4	.35	.27	.21	.17	88.15	8069.03	0.00*

Notice that the minimum index of utility(*) which occurs at corner boundary point 4 is of zero value at the trial rate of return of 21.85% for an angular coefficient of -0.05.

C. Determination of Final Angular Coefficient

In the context of incomplete knowledge, there is a range of utility index ROR for each value of the angular coefficient. The approach to identify the maximum and minimum utility index ROR was presented in Section B. This section will focus on the selection of the angular coefficient which will form the final decision line.

Before determining the value of the final angular coefficient, the possible angular coefficients should be evaluated for each pair of alternatives. For each pair of alternatives, i.e., alternatives compared with present conditions or compared with one another, a line of indiscernibility is drawn between the highest rate of return and the lowest rate of return representing the "best" and "worst" states of nature. Since a rate of return higher than the highest or lower than the lowest is impossible, a reasonable utility index ROR must fall within the line of indiscernibility. Therefore, those angular coefficients, which could cause the utility index ROR to be greater than the highest rate of return from the "best" state of nature or less than the lowest rate of return from the "worst" state of nature, need not to be considered any further. Those angular coefficients, which could cause the utility index ROR to fall within the line of indiscernibility, are the candidates for the final angular coefficient.

The minimum angular coefficient, $\underline{a}_{min}$, is the lower limit of the possible angular coefficients at which the minimum utility index ROR are greater than or equal to the smallest rate of return on the line of indiscernibility. It is not necessary to consider an angular coefficient less than $\underline{a}_{min}$ because any angular coefficient less than $\underline{a}_{min}$ could cause a utility index ROR lower than the lowest rate of return possible.

The maximum angular coefficient, $\underline{a}_{max}$, is the upper limit of the possible angular coefficients at which the maximum utility index ROR are less than or equal to the largest rate of return on the line of indiscernibility. Theoretically, those angular coefficients between $\underline{a}_{min}$ and $\underline{a}_{max}$ are all possible for final selection. However, since the value of $\underline{a}_{max}$ is always positive, causing contradictory conclusions as discussed in Chapter V, the possible range for the angular coefficient is from $\underline{a}_{min}$ to zero.

Although the value of the minimum angular coefficient for a certain pair of alternatives can be found by the trial and error method as in Constant's decision approach, a more straightforward approach to locate the value of the minimum angular coefficient is developed. The approach developed under weak ranking is also applicable under strict ranking.

The payoff values for all possible states of nature are first calculated according to i_{min}. If there are more than two distinct payoff values, the minimum index of utility must occur at one of the corner boundary points. The minimum angular coefficient must also occur at one of the corner boundary points. The possible minimum angular coefficient for each corner boundary point can be obtained by the following equation,

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 7-6)}$$

The procedure is to choose the largest $\underline{a}_{min}$ value from among the $\underline{a}_{min}$ values of all corner boundary points as the minimum angular coefficient for this pair of alternatives. The largest $\underline{a}_{min}$ is selected because it is the most conservative angular coefficient at which the minimum utility index ROR is equal to i_{min}.

If there are only two distinct payoff values, it is possible that the minimum index of utility exists inside the feasible region. The coefficient of risk aversion can be calculated according to the following equation,

$$b = \frac{-(X_t+X_1) \pm 2\sqrt{X_tX_1}}{(X_t-X_1)^2} \qquad \text{(Eq. 7-7)}$$

Notice that there are two values of the coefficient of risk aversion obtained from Eq. 7-7. However, only the value of the coefficient of risk aversion that satisfies all of the following tests should be used to determine the minimum angular coefficient for this pair of alternatives.

Test 1: The value of the coefficient of risk aversion must be negative.

Test 2: The necessary conditions: $-1/2 + C'' \leq 1/2b(X_t-X_1) \leq 1/2 - C'$ (Eq. 7-4) must be satisfied.

Test 3: At least one basic solution of Eqs. 7-1 and 7-2 must be feasible.

If both of the values of the coefficient of risk aversion obtained from Eq. 7-7 fail to satisfy the tests, the global minimum index of utility must occur at one of the corner boundary points. The minimum angular coefficient must also occur at the corner boundary points.

After the possible ranges for all pairs of alternatives have been found, the common range of the angular coefficients can be identified.

The lower limit of the common range of the angular coefficients is the maximum of the minimum angular coefficients, max($\underline{a}_{min}$), for all pairs of alternatives. The upper limit of the common range of the angular coefficients is zero. The common range of the angular coefficients represents the assembly of all possible values of angular coefficients from which a decision maker can choose. The lower limit of the common range of the angular coefficients forms the final decision line. If the pair of alternatives on which the final angular coefficient is established is not included on the final decision line, adjustment of the selected final angular coefficient is necessary.

D. Summary

For strict ranking in the context of incomplete knowledge, the necessary steps in the simplified approach are restated as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

2. For all pairs of alternatives, i.e., alternative compared with present conditions and with one another, draw a line of indiscernibility between the lowest rate of return and the highest rate of return representing the "best case" and "worst case" scenarios. Let the lowest rate of return ("worst case" scenario) be defined as i_{min}, and the highest rate of return ("best case" scenario) as i_{max}.

3. Obtain the minimum angular coefficients based on the value of i_{min} on the lines of indiscernibility according to either one of the following cases:

   a) If there are more than two distinct payoff values, calculate $\underline{a}_{min}$ for all the corner boundary points according to Eq. 7-6,

$$\underline{a}_{j,min} = \frac{-\ EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 7-6)}$$

From among the $\underline{a}_{min}$ values of all corner boundary points, choose the largest (least negative) $\underline{a}_{min}$ value as the minimum angular coefficient for each pair of alternatives.

b) If there are only two distinct payoff values, calculate the possible values of b, the coefficient of risk aversion.

$$b = \frac{-(X_t+X_1) \pm 2\ \sqrt{X_t X_1}}{(X_t-X_1)^2} \qquad \text{(Eq. 7-7)}$$

Only the value of the coefficient of risk aversion that satisfies all of the following tests should be used to determine the minimum angular coefficient for this pair of alternatives.

Test 1: The value of the coefficient of risk aversion must be negative.

Test 2: The necessary conditions: $-\ 1/2 + C'' \leq 1/2b(X_t-X_1) \leq 1/2 - C'$ (Eq. 7-4) must be satisfied.

Test 3: At least one basic solution of Eqs. 7-1 and 7-2 must be feasible.

If both of the values of the coefficient of risk aversion obtained from Eq. 7-7 fail to satisfy the tests, the minimum angular coefficient must also occur at the corner boundary points.

4. For all pairs of alternatives, choose the largest (least negative) value of $\underline{a}_{j,min}$ calculated in Step 3. The range for possible final $\underline{a}$ values is then defined by the maximum of $\underline{a}_{min}$ (which is a negative value) and zero. The max($\underline{a}_{min}$) value is the most conservative angular coefficient within the possible range; as the angular coefficients within the range becomes less negative, decisions become less conservative.

5. Calculate the minimum utility index ROR for each pair of alternatives based on the angular coefficient selected using a trial and error routine. In each trial and error routine, the payoff values for various states of nature are calculated.

a) If there are more than two distinct payoff values, it is only necessary to calculate the values of index of utility at corner boundary points in order to determine the minimum index of utility.

b) If there are only two distinct payoff values, it is possible that a relative minimum index of utility exists inside the feasible region. Check the necessary conditions: $-1/2 + C'' \leq 1/2b(X_t - X_1) \leq 1/2 - C'$ (Eq. 7-4). If the necessary conditions are not met, the global minimum index of utility must occur at the corner boundary points. If the necessary conditions are met, Eqs. 7-1, 7-2, and 7-3 are served to locate the multiple relative minimum indexes of utility. Check all basic solutions of Eqs. 7-1 and 7-2. If at least one basic solution is feasible, the global minimum index of utility can be calculated by Eq. 7-5. If none of the basic solutions is feasible, the global minimum index of utility must occur at the corner boundary points.

The trial and error routine is repeated until a utility index ROR is found at which the minimum index of utility is of zero value.

6. Form the final decision line by taking the minimum utility index ROR for each pair of alternatives calculated in Step 5.

   Examine the final decision line. If the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is included on the final decision line, the decision procedure is completed. However, if the pair of alternatives from which the final $\underline{a}_{min}$ value is selected is not included on the final decision line, the final $\underline{a}_{min}$ value must be modified. From among the remaining pairs of alternatives, take the next least negative value of $\underline{a}_{min}$ as the revised final angular coefficient. Steps 5 and 6 are repeated as many times as necessary.

The same numerical example used for Constant's approach will be presented again to illustrate the formation of the final decision line under strict ranking. The necessary steps are as follows:

1. For each state of nature, solve for the rates of return comparing each alternative with present conditions and with one another.

   For state of nature N_1, the rates of return are as follows:

	Internal ROR over			
	P	A	B	C
Alternative A	22.11%			
B	21.15%	18.72%		
C	20.42%	17.57%	15.24%	
D	19.06%	15.94%	14.05%	13.46%

For state of nature N_2, the rates of return are as follows:

	Internal ROR over			
	P	A	B	C
Alternative A	23.68%			
B	23.57%	23.30%		
C	21.63%	18.12%	5.91%	
D	21.08%	18.45%	15.03%	18.93%

For state of nature N_3, the rates of return are as follows:

	Internal ROR over			
	P	A	B	C
Alternative A	21.96%			
B	23.57%	27.33%		
C	19.22%	14.43%	-48.22%	
D	20.38%	18.81%	12.40%	24.61%

For state of nature N_4, the rates of return are as follows:

	Internal ROR over			
	P	A	B	C
Alternative A	25.08%			
B	22.36%	14.99%		
C	24.04%	22.29%	34.26%	
D	20.99%	16.71%	17.79%	7.11%

2. For all pairs of alternatives, i.e., alternative compared with present conditions or compared with one another, draw a line of indiscernibility between the highest rate of return and the lowest rate of return representing the "best case" and "worst case" scenarios, respectively. Let the highest rate of return be defined as i_{max}, and the lowest rate of return as i_{min}. The lines of indiscernibility for all pairs of alternatives on the decision line are as follows:

```
A - P +-------------------------------------------******-----------
                                           21.96%  25.08%
```

```
B - P +-----------------------------------------*****--------------
                                          21.15% 23.57%

B - A +-------------------------**************************--------
                          14.99%                    27.33%

C - P +-------------------------------**********------------
                                    19.22%   24.04%

C - A +-----------------------------****************----------------
                          14.43%           22.29%

C - B **************************************************************
       0                                                      34.26%

D - P +----------------------------------*****------------------
                                    19.06%  21.08%

D - A +------------------------------******---------------------
                              15.94%   18.81%

D - B +-----------------------************-------------------------
                        12.40%      17.99%

D - C +-----***********************************************-------------
           7.11%                                     24.61%
```

3. Obtain the minimum angular coefficients based on the value of i_{min} on the lines of indiscernibility.

   For alternative A compared with present conditions, the value of i_{min} on the line of indiscernibility is found to be 21.96%. The following figures of merit (AEX in this case) can be calculated based on 21.96%:

   N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = 11.05$
   N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 130.49$
   N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = 0.00$
   N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 240.19$

   Since there are more than two distinct payoff values, calculate $\underline{a}_{min}$ for all the corner boundary points according to Eq. 7-6,

$$\underline{a}_{j,min} = \frac{-\,EXP[AEX]_j}{(i_{j,min})(VAR[AEX]_j)} \qquad \text{(Eq. 7-6)}$$

   The resulting values of the $\underline{a}_{min}$ for each corner boundary point are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	a_{min} value
1	.80	.12	.06	.02	29.31	2436.06	-0.05479
2	.50	.42	.06	.02	65.14	4123.79	-0.07193
3	.40	.32	.26	.02	50.98	4052.37	-0.05730
4	.35	.27	.21	.17	79.93	8058.20	-0.04517

Choose the largest (least negative) a_{min} value from among the a_{min} values of all corner boundary points, -0.04517, as the minimum angular coefficient for alternative A compared with present conditions.

In the same manner, the minimum angular coefficients for all pairs of alternatives at various corner boundary points (CBP) can be calculated and listed as follows,

Pair of Alternatives	a_{min} CBP 1	CBP 2	CBP 3	CBP 4	Max.
A - P	-.05479	-.07193	-.05730	-.04517	-.04517
B - P	-.02341	-.03670	-.04568	-.05087	-.02341
B - A	-.14576	-.15299	-.11002	-.07055	-.07055
C - P	-.11243	-.10308	-.05351	-.03244	-.03244
C - A	-.43840	-.43459	-.14537	-.08761	-.08761
C - B	+	+	+	+	+
D - P	-.02353	-.03543	-.04658	-.05208	-.02353
D - A	-.04097	-.06620	-.07791	-.07825	-.04097
D - B	-.57522	-.53655	-.23322	-.13765	-.13765
D - C	-.28318	-.29328	-.19360	-.11676	-.11676

4. From among all pairs of alternatives, choose the least negative value of $a_{j,min}$, -0.02341 (B over P). This value is the most conservative angular coefficient within the possible range from -0.02341 to zero.

5. Calculate the minimum utility index ROR for each pair of alternatives based on the angular coefficient selected using a trial and error routine. In each trial and error routine, the payoff values for the various states of nature are calculated according to the trial rate of return.

   For alternative A compared to present conditions, the four payoff values (AEX) for the various states of nature using a trial rate of return of 22.18% are as follows:

N_1: $e_{F1} = 0\%$, $e_{F2} = 0\%$, $AEX_1 = -5.62$
N_2: $e_{F1} = 2\%$, $e_{F2} = 1\%$, $AEX_2 = 113.66$
N_3: $e_{F1} = 2\%$, $e_{F2} = -1\%$, $AEX_3 = -16.66$
N_4: $e_{F1} = 1\%$, $e_{F2} = 3\%$, $AEX_4 = 223.20$

Since all the four payoff values are different from one another, the extreme values of the index of utility can be found at the corner boundary points. Using the angular coefficient of -0.02341, the values of the mean, variance, and index of utility at each of the corner boundary points are listed as follows,

Corner boundary point	Probability for state of nature 1	2	3	4	Mean	Variance	Index of utility
1	.80	.12	.06	.02	12.61	2429.45	0.00*
2	.50	.42	.06	.02	48.39	4112.63	27.04
3	.40	.32	.26	.02	34.26	4041.46	13.27
4	.35	.27	.21	.17	63.17	8036.23	21.44

Since the minimum index of utility(*) is zero at the trial rate of return of 22.18%, this trial rate of return is the minimum utility index ROR for alternative A compared to present conditions at the angular coefficient of -0.02341.

In the same manner, the minimum utility index ROR for all pairs of alternatives can be calculated and summarized as follows,

	Minimum utility index ROR over P	A	B	C
Alternative A	22.18%			
B	21.15%	18.83%		
C	19.73%	16.58%	2.34%	
D	19.06%	16.15%	13.85%	13.36%

6. The final decision line based on the angular coefficient selected, i.e., -0.02341, is as follows,

```
                D                           B          A        P
+----------------------------------+-----------+--------+-------
0                               13.85%      18.83%   22.18%
```

Since the final decision line does not include the pair of alternatives B over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.02353 (D over P).

5a. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.02353.

	Minimum utility index ROR over P	A	B	C
Alternative A	22.18%			
B	21.14%	18.82%		
C	19.72%	16.58%	2.34%	
D	19.06%	16.15%	13.85%	13.35%

6a. The final decision line based on the angular coefficient selected, i.e., -0.02353, is as follows,

```
                  D                    B          A        P
+--------------------------------+----------+---------+--------
0                              13.85%     18.82%    22.18%
```

Since the final decision line does not include the pair of alternatives D over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.03244 (C over P).

5b. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.03244.

	Minimum utility index ROR over P	A	B	C
Alternative A	22.12%			
B	20.97%	17.99%		
C	19.22%	16.41%	2.00%	
D	18.93%	16.04%	13.78%	12.38%

6b. The final decision line based on the angular coefficient selected, i.e., -0.03669, is as follows,

```
                  D                    B          A        P
+--------------------------------+----------+---------+--------
0                              13.78%     17.99%    22.12%
```

Since the final decision line does not include the pair of alternatives C over P, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.04097 (D over A).

5c. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.04097.

	Minimum utility index ROR over			
	P	A	B	C
Alternative A	22.05%			
B	20.80%	17.24%		
C	18.76%	16.15%	1.76%	
D	18.80%	15.94%	13.72%	11.57%

6c. The final decision line based on the angular coefficient selected, i.e., -0.04097, is as follows,

```
                 D                         B          A          P
+--------------------------------+----------+-----------+--------
0                             13.72%     17.24%      22.05%
```

Since the final decision line does not include the pair of alternatives D over A, which provided the bases for the selection of the final angular coefficient, the final angular coefficient must be modified. From among the remaining pairs of alternatives, choose the next least negative value of $\underline{a}_{j,min}$, -0.04517 (A over P).

5d. Calculate the minimum utility index ROR based on the angular coefficient selected, i.e., -0.04517.

	Minimum utility index ROR over			
	P	A	B	C
Alternative A	21.96%			
B	20.72%	16.89%		
C	18.53%	15.98%	1.66%	
D	18.73%	15.89%	13.69%	11.20%

6d. The final decision line based on the angular coefficient selected, i.e., -0.04517, is as follows,

```
                 D                         B          A          P
+--------------------------------+----------+-----------+--------
0                             13.69%     16.89%      21.96%
```

Since the final decision line does include the pair of alternatives A over P, the final decision line is completed.

VIII. CONCLUSION

The research objectives stated in the introduction require a complete and final decision line when evaluating a set of mutually exclusive alternatives under the context of incomplete knowledge for both weak and strict ranking.

Chapter III disclosed the procedures of searching for the extreme variances of payoffs for alternatives under strict ranking. Since the extreme variances are a measure of dispersion of the expected payoffs, it may be interpreted as a measure of risk attached to each alternative under conditions of strict ranking in the context of incomplete knowledge.

Chapter IV explored how the variances and the expected values of payoffs can be combined into an index of utility with a coefficient of risk aversion. In order to reduce the chances of unwanted results, a negative coefficient of risk aversion was used to apply heavier penalties for greater variances. With the coefficient of risk aversion assumed to be known, an algorithm was developed to search for the extreme indexes of utility under conditions of strict ranking in the context of incomplete knowledge.

In Chapter V, the coefficient of risk aversion was expressed in terms of the minimum attractive rates of return and an angular coefficient. A method to determine the appropriate value of the angular coefficient for one set of mutually exclusive alternative was developed under the context of uncertainty. This method was then modified and was successfully applied to both the context of uncertainty and the context of risk.

Chapters VI and VII extended the modified technique of finding the appropriate angular coefficient to conditions of weak ranking and strict ranking, respectively. After the angular coefficient was determined, it became possible to find the extreme rates of return on index of utility by applying the algorithm established in Chapter IV to search for the extreme indexes of utility. A complete and final decision line was then constructed based on the minimum rates of return on index of utility.

Further research is recommended at this point. Sensitivity analysis of the basic results should be conducted. The sensitivity, and thus the importance, of each variable should be determined. Also, a sensitivity analysis of the final decision line should be conducted under conditions of both weak and strict ranking to ascertain the value of additional information of the probabilities of possible states of nature.

The utility function adopted in this research is a linear combination of the expected value and the variance. In view of the recent developments in non-linear utility theory, it would be of interest to apply a non-linear utility function in the decision procedure. Since several curvilinear functions could describe the relationship between the minimum attractive rates of return and the coefficient of risk aversion, it is important to investigate the theoretical function that best describes the decision process.

It is also noted that equal weights have been conventionally given to both positive and negative deviations when using the variance as a measure of dispersion. The feasibility of using asymmetric measures of dispersion, where more weight is given to the negative deviation than to the positive deviation, needs to be studied.

The final decision line for one set of mutually exclusive alternatives can be determined as a result of this research. The possibility of developing a decision procedure for a combination of independent and mutually exclusive alternatives under the context of risk, uncertainty, and incomplete knowledge should be investigated.

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